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Trigonometrically Fitted Block Backward Differentiation Methods for First Order Initial Value Problems with Periodic Solution

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

A family of Trigonometrically Fitted Backward Differentiation Formula (TBDF) whose coefficients depend on the frequency and step size for periodic initial value problems is presented. The method is constructed based on collocation techniques. The primary method of TBDF is obtained from its continuous version while the additional methods are derived from the derivative of the continuous version which are combined and applied in block form as simultaneous numerical integrators. The stability properties of the method are discussed and numerical experiments show that the TBDF is an accurate numerical integrator.

Keywords: Backward differentiation formula; collocation; continuous form; frequency; periodic initial value problem; stability; trigonometrically-fitted.

1 Introduction

One of the most popular classes of multistep methods for solving Ordinary Differential Equation particularly stiff is the Backward Differentiation formula (BDF). This method was first used for the solution of stiff

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problems by Curtis and Hirschfelder [1]. Over the years several variants of BDF have been developed and discussed in the literature (see [2]-[9]) for more details and are referenced therein. This study focuses on the numerical integration of the first order Initial Value Problem of the form

$$y' = f(x, y), y(x_0) = y_0, \tag{1}$$

whose solution is periodic and f satisfies the Lipschitz condition of the existence and uniqueness of the solution. This problems often exist in a number of applied science such as Electronics, Chemical Kinetics, Theoretical Chemistry, Medical Science and Control Theory.

Our goal in this paper is to adapt Backward Differentiation Formula with trigonometric coefficients to numerically integrate (1) in block by block fashion. There have been a number of step by step methods based on trigonometric polynomials for solving (1) or systems of (1) (see [10]-[13]). Some of these methods are sensitive to changes in the frequency ω, some require the eigenvalues of the Jacobian to be purely imaginary and some are expensive to implement. To overcome these set back the block methods are proposed. The idea of block methods started with Milne [14], who used this idea only as a mean of generating starting values for predictor-corrector algorithms. Brugnano and Trigiante [15] developed the block LMM into methods for solving IVPs. Block methods contain two parts viz: main and complementary methods [16]. Some of the advantages of block methods include but not limited to permission of easy change of step length [17], self-starting and thus avoiding the use of other method(s) to get the starting solution, overcoming the overlapping of pieces of solution and obtaining numerical solution at more than one point at a time [18].

2 Construction of TBDF

This section discusses the procedure for the construction of Trigonometrically-Fitted Backward Differentiation Formula (TBDF) with one main method and $(k-1)$ additional methods for $k = 2(1)4$. The $k-$ step TBDF has the form

$$\sum_{j=0}^{k} \alpha_j(u) y_{n+j} = h\beta_k(u) f_{n+k}, \tag{2}$$

Where $u = \omega h, \omega$ is frequency, α_j and β_k which depends on frequency and step size are parameters to be determined distinctively. Also, y_{n+j} is the numerical approximation to exact solution $y(x_{n+j})$ and $f_{n+k} = f(x_{n+k}, y_{n+k})$. We first obtain the continuous approximation for TBDF via multistep collocation technique with the assumption that the exact solution $y(x)$ is approximated on the interval $[x_0, x_n]$ by the interpolating function $\Gamma(x)$ given by

$$\tau(x) = \sum_{j=0}^{k-1} \alpha_j(x,u) y_{n+j} + h\beta_k(x,u) f_{n+k}. \tag{3}$$

We demand that the following conditions be imposed on equation (3)

$$\Gamma(x_{n+j}) = y_{n+j}, j = 0(1)(k-1), \tag{4}$$

$$\Gamma'(x)\Big|_{x=x_{n+j}} = f_{n+j}, j = k. \tag{5}$$

Equations (4) and (5) lead to a system of $(k+1)$ equations which in this paper is solved using Maple 2016.2. The exploration of CAS becomes necessary because the derivation by hand becomes more difficult for $k \geq 2$. The continuous form of TBDF is obtained by substituting the values of $a_j, j = 0(1)k$ into equation (3). After some algebraic manipulations, the continuous TBDF is expressed as

$$\tau(x) = \sum_{j=0}^{k-1} \alpha_j(x,u) y_{n+j} + h\beta_k(x,u) f_{n+k} . \tag{6}$$

Differentiating equation (6) with respect to x , to obtain

$$\tau^{'}(x) = \frac{1}{h} \sum_{j=0}^{k-1} \overline{\alpha_{j,1}}(x,u) y_{n-j} + \overline{\beta_{k,1}}(x,u) f_{n+j} . \tag{7}$$

The primary method of TBDF is obtained by evaluating equation (6) at $x = x_{n+k}$ while the $(k-1)$ additional methods are obtained by evaluating equation (7) at $x = x_{n+j}, j = 1,2,3...(k-1)$ We emphasis that the coefficients of equations (6) and (7) are in trigonometric form. To avoid the substantial cancellations which might occur when h is small, the use of the power series expansion of the parameters is preferable [19]. The coefficients and the power series expansions of the primary method and additional methods of TBDF up to $O(u^8)$ are as listed.

For $k = 2$

$$h f_{n+1} = -\frac{h \sin(u)\, u\, y_n}{-\cos(2u) + \cos(u)} + \frac{h \sin(u)\, u\, y_{n+1}}{-\cos(2u) + \cos(u)} + \frac{h(-h\cos(u) + h) f_{n+2}}{-\cos(2u) + \cos(u)}$$

$$y_{n+2} = \frac{(-u\cos(u) + u) y_n}{u\cos(2u) - u\cos(u)} + \frac{(u\cos(2u) - u) y_{n+1}}{u\cos(2u) - u\cos(u)} + \frac{(-2h\sin(u) + h\sin(2u)) f_{n+2}}{u\cos(2u) - u\cos(u)}$$

$$h f_{n+1} = \left(-\frac{2}{3} - \frac{1}{6}u^2 - \frac{13}{360}u^4 - \frac{121}{15120}u^6 - \frac{1093}{604800}u^8\right) y_n + \left(\frac{2}{3} + \frac{1}{6}u^2 + \frac{13}{360}u^4 + \frac{121}{15120}u^6 + \frac{1093}{604800}u^8\right) y_{n+1} + \left(\frac{1}{3}h + \frac{1}{9}hu^2 + \frac{1}{36}hu^4 + \frac{7}{1080}hu^6 + \frac{809}{544320}hu^8\right) f_{n+2}$$

$$y_{n+2} = \left(-\frac{1}{3} - \frac{1}{9}u^2 - \frac{1}{36}u^4 - \frac{7}{1080}u^6 - \frac{809}{544320}u^8\right) y_n + \left(\frac{4}{3} + \frac{1}{9}u^2 + \frac{1}{36}u^4 + \frac{7}{1080}u^6 + \frac{809}{544320}u^8\right) y_{n+1} + \left(\frac{2}{3}h + \frac{1}{9}hu^2 + \frac{13}{540}hu^4 + \frac{41}{7560}hu^6 + \frac{671}{544320}hu^8\right) f_{n+2}$$

For $k = 3$

$$h f_{n+1} = \frac{(u^2\sin(2u) + u\cos(2u) - u) y_n}{u\cos(3u) - 2u\cos(2u) + u\cos(u) + 2\sin(u) - \sin(2u)} + \frac{(-2u^2\sin(2u) + u\cos(u) - u\cos(3u)) y_{n+1}}{u\cos(3u) - 2u\cos(2u) + u\cos(u) + 2\sin(u) - \sin(2u)} + \frac{(u^2\sin(2u) - u\cos(u) + u\cos(3u) - u\cos(2u) + u) y_{n+2}}{u\cos(3u) - 2u\cos(2u) + u\cos(u) + 2\sin(u) - \sin(2u)} + \frac{(2\cos(u)\, h u + 2\sin(u)\, h - \sin(2u)\, h - 2hu) f_{n+3}}{u\cos(3u) - 2u\cos(2u) + u\cos(u) + 2\sin(u) - \sin(2u)}$$

$$h f_{n+2} = \frac{\left(u^2 \sin(u) - 2u\cos(u) + u\cos(2u) + u\right) y_n}{u\cos(3u) - 2u\cos(2u) + u\cos(u) + 2\sin(u) - \sin(2u)} + \frac{\left(-2u^2\sin(u) + u\cos(u) - u\cos(3u) + u\cos(2u) - u\right) y_{n+1}}{u\cos(3u) - 2u\cos(2u) + u\cos(u) + 2\sin(u) - \sin(2u)} + \frac{\left(u^2\sin(u) + u\cos(u) + u\cos(3u) - 2u\cos(2u)\right) y_{n+2}}{u\cos(3u) - 2u\cos(2u) + u\cos(u) + 2\sin(u) - \sin(2u)} + \frac{\left(-2\cos(u)\,hu + \cos(2u)\,hu + 2\sin(u)\,h - \sin(2u)\,h + hu\right) f_{n+3}}{u\cos(3u) - 2u\cos(2u) + u\cos(u) + 2\sin(u) - \sin(2u)}$$

$$y_{n+3} = \frac{\left(-2u\cos(u) + u\cos(2u) + 2\sin(u) - \sin(2u) + u\right) y_n}{u\cos(3u) - 2u\cos(2u) + u\cos(u) + 2\sin(u) - \sin(2u)} + \frac{\left(3u\cos(u) - u\cos(3u) - \sin(u) + \sin(3u) - \sin(2u) - 2u\right) y_{n+1}}{u\cos(3u) - 2u\cos(2u) + u\cos(u) + 2\sin(u) - \sin(2u)} + \frac{\left(2u\cos(3u) - 3u\cos(2u) + \sin(u) - \sin(3u) + \sin(2u) + u\right) y_{n+2}}{u\cos(3u) - 2u\cos(2u) + u\cos(u) + 2\sin(u) - \sin(2u)} + \frac{\left(5\sin(u)\,h + \sin(3u)\,h - 4\sin(2u)\,h\right) f_{n+3}}{u\cos(3u) - 2u\cos(2u) + u\cos(u) + 2\sin(u) - \sin(2u)}$$

$$h f_{n+1} = \left(-\frac{4}{11} - \frac{116}{1815}u^2 - \frac{76999}{8385300}u^4 - \frac{67724}{49413375}u^6 - \frac{1828868309}{8522818920000}u^8\right) y_n + \left(-\frac{4}{11} + \frac{181}{1815}u^2 + \frac{105359}{8385300}u^4 + \frac{4716209}{2767149000}u^6 + \frac{2166444199}{8522818920000}u^8\right) y_{n+1} + \left(\frac{8}{11} - \frac{13}{363}u^2 - \frac{1418}{419265}u^4 - \frac{184733}{553429800}u^6 - \frac{33757589}{852281892000}u^8\right) y_{n+2} + \left(-\frac{1}{11}h - \frac{17}{605}hu^2 - \frac{16213}{2795100}hu^4 - \frac{956293}{922383000}hu^6 - \frac{497097473}{2840939640000}hu^8\right) f_{n+3}$$

$$h f_{n+2} = \left(\frac{5}{22} + \frac{211}{3630}u^2 + \frac{22258}{2096325}u^4 + \frac{613439}{345893625}u^6 + \frac{1234004941}{4261409460000}u^8\right) y_n + \left(-\frac{14}{11} - \frac{251}{3630}u^2 - \frac{209059}{16770600}u^4 - \frac{11397739}{5534298000}u^6 - \frac{5693483879}{17045637840000}u^8\right) y_{n+1} + \left(\frac{23}{22} + \frac{4}{363}u^2 + \frac{6199}{3354120}u^4 + \frac{316543}{1106859600}u^6 + \frac{151492823}{3409127568000}u^8\right) y_{n+2} + \left(\frac{2}{11}h + \frac{57}{1210}hu^2 + \frac{16341}{1863400}hu^4 + \frac{914701}{614922000}hu^6 + \frac{464283961}{1893959760000}hu^8\right) f_{n+3}$$

$$y_{n+3} = \left(\frac{2}{11} + \frac{57}{1210}u^2 + \frac{16341}{1863400}u^4 + \frac{914701}{614922000}u^6 + \frac{464283961}{1893959760000}u^8\right) y_n + \left(-\frac{9}{11} - \frac{21}{605}u^2 - \frac{8313}{931700}u^4 - \frac{503593}{307461000}u^6 - \frac{262293373}{946979880000}u^8\right) y_{n+1} + \left(\frac{18}{11} - \frac{3}{242}u^2 + \frac{57}{372680}u^4 + \frac{18497}{122984400}u^6 + \frac{12060557}{378791952000}u^8\right) y_{n+2} + \left(\frac{6}{11}h + \frac{36}{605}hu^2 + \frac{2007}{232925}hu^4 + \frac{34259}{25621750}hu^6 + \frac{16832549}{78914990000}hu^8\right) f_{n+3}$$

For $k=4$

$$
\begin{aligned}
hf_{n+1} &= \frac{\left(2\sin(3u)\,u^2+3u\cos(3u)-7u\cos(u)+u\cos(2u)-6\sin(2u)+12\sin(u)+3u\right)y_n}{-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)}\\
&+\frac{\left(-6\sin(3u)\,u^2-3u\cos(4u)+12u\cos(u)-9u\cos(2u)-6\sin(2u)-6\sin(u)+6\sin(3u)\right)y_{n+1}}{-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)}\\
&+\Big(\big(6\sin(3u)\,u^2-3u\cos(3u)+4u\cos(4u)+7u\cos(u)+7u\cos(2u)+6\sin(2u)+6\sin(u)-6\sin(3u)\\
&-15u\big)\,y_{n+2}\Big)\Big/\big(-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)\\
&-5\sin(3u)\big)+\frac{\left(-2\sin(3u)\,u^2-u\cos(4u)-12u\cos(u)+u\cos(2u)+6\sin(2u)-12\sin(u)+12u\right)y_{n+3}}{-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)}\\
&+\frac{\left(8\cos(u)\,hu-2\cos(2u)\,hu-4h\sin(2u)+5h\sin(u)+\sin(3u)\,h-6uh\right)f_{n+4}}{-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)}
\end{aligned}
$$

$$
\begin{aligned}
hf_{n+2} &= \frac{\left(2\sin(2u)\,u^2+u\cos(3u)+7u\cos(u)-4\sin(2u)+8\sin(u)-8u\right)y_n}{-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)}\\
&+\frac{\left(-6\sin(2u)\,u^2-u\cos(4u)-8u\cos(u)-6u\cos(2u)-4\sin(2u)-4\sin(u)+4\sin(3u)+15u\right)y_{n+1}}{-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)}\\
&+\frac{\left(6\sin(2u)\,u^2+3u\cos(3u)-11u\cos(u)+8u\cos(2u)+4\sin(2u)+4\sin(u)-4\sin(3u)\right)y_{n+2}}{-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)}\\
&+\frac{\left(-2\sin(2u)\,u^2-4u\cos(3u)+u\cos(4u)+12u\cos(u)-2u\cos(2u)+4\sin(2u)-8\sin(u)-7u\right)y_{n+3}}{-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)}\\
&+\frac{\left(-8\cos(u)\,hu+2\cos(2u)\,hu+4h\sin(2u)-5h\sin(u)-\sin(3u)\,h+6uh\right)f_{n+4}}{-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)}
\end{aligned}
$$

$$
\begin{aligned}
hf_{n+3} &= \frac{\left(2\sin(u)\,u^2-u\cos(3u)-11u\cos(u)+7u\cos(2u)-2\sin(2u)+4\sin(u)+5u\right)y_n}{-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)}+\Big(\big(\\
&-6\sin(u)\,u^2-3u\cos(3u)+u\cos(4u)+23u\cos(u)-9u\cos(2u)-2\sin(2u)-2\sin(u)+2\sin(3u)-12u\big)\\
&y_{n+1}\Big)\Big/\big(-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)\big)\\
&+\Big(\big(6\sin(u)\,u^2+17u\cos(3u)-4u\cos(4u)-5u\cos(u)-15u\cos(2u)+2\sin(2u)+2\sin(u)-2\sin(3u)\\
&+7u\big)\,y_{n+2}\Big)\Big/\big(-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)\big)\\
&+\frac{\left(-2\sin(u)\,u^2-13u\cos(3u)+3u\cos(4u)-7u\cos(u)+17u\cos(2u)+2\sin(2u)-4\sin(u)\right)y_{n+3}}{-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)}\\
&+\frac{\left(2\cos(3u)\,hu+6\cos(u)\,hu-6\cos(2u)\,hu+12h\sin(2u)-15h\sin(u)-3\sin(3u)\,h-2uh\right)f_{n+4}}{-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)}
\end{aligned}
$$

$$
\begin{aligned}
y_{n+4} &= \frac{\left(-2u\cos(3u)-6u\cos(u)+6u\cos(2u)-12\sin(2u)+15\sin(u)+3\sin(3u)+2u\right)y_n}{-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)}\\
&+\frac{\left(2u\cos(4u)+16u\cos(u)-12u\cos(2u)+14\sin(2u)-28\sin(u)+4\sin(3u)-3\sin(4u)-6u\right)y_{n+1}}{-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)}\\
&+\frac{\left(12u\cos(3u)-6u\cos(4u)-12u\cos(u)+16\sin(2u)+8\sin(u)-24\sin(3u)+8\sin(4u)+6u\right)y_{n+2}}{-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)}\\
&+\frac{\left(-16u\cos(3u)+6u\cos(4u)+12u\cos(2u)+2\sin(2u)-20\sin(u)+12\sin(3u)-5\sin(4u)-2u\right)y_{n+3}}{-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)}\\
&+\frac{\left(28h\sin(2u)-28h\sin(u)-12\sin(3u)\,h+2\sin(4u)\,h\right)f_{n+4}}{-2u\cos(u)+6u\cos(2u)-6u\cos(3u)+2u\cos(4u)-25\sin(u)+20\sin(2u)-5\sin(3u)}
\end{aligned}
$$

$$hf_{n+1}=\left(-\frac{13}{50}-\frac{281}{7500}u^2-\frac{65833}{15750000}u^4-\frac{1138471}{2362500000}u^6-\frac{2156016847}{36382500000000}u^8\right)y_n+\left(-\frac{39}{50}+\frac{239}{2500}u^2+\frac{49927}{5250000}u^4+\frac{777649}{787500000}u^6+\frac{1381606993}{12127500000000}u^8\right)y_{n+1}+\left(\frac{69}{50}-\frac{169}{2500}u^2-\frac{23417}{5250000}u^4-\frac{176279}{787500000}u^6-\frac{12989129}{1732500000000}u^8\right)y_{n+2}+\left(-\frac{17}{50}+\frac{71}{7500}u^2-\frac{13697}{15750000}u^4-\frac{665639}{2362500000}u^6-\frac{245147489}{5197500000000}u^8\right)y_{n+3}+\left(\frac{1}{25}h+\frac{7}{625}hu^2+\frac{2651}{1312500}hu^4+\frac{8591}{28125000}hu^6+\frac{129068309}{3031875000000}hu^8\right)f_{n+4}$$

$$hf_{n+2}=\left(\frac{7}{75}+\frac{13}{625}u^2+\frac{1046}{328125}u^4+\frac{14293}{32812500}u^6+\frac{2718304}{47373046875}u^8\right)y_n+\left(-\frac{18}{25}-\frac{57}{1250}u^2-\frac{5717}{875000}u^4-\frac{111379}{131250000}u^6-\frac{72958201}{673750000000}u^8\right)y_{n+1}+\left(\frac{3}{25}+\frac{11}{625}u^2+\frac{487}{328125}u^4+\frac{8363}{98437500}u^6+\frac{201469}{108281250000}u^8\right)y_{n+2}+\left(\frac{38}{75}+\frac{9}{1250}u^2+\frac{1629}{875000}u^4+\frac{129169}{393750000}u^6+\frac{42485519}{866250000000}u^8\right)y_{n+3}+\left(-\frac{1}{25}h-\frac{7}{625}hu^2-\frac{2651}{1312500}hu^4-\frac{8591}{28125000}hu^6-\frac{129068309}{3031875000000}hu^8\right)f_{n+4}$$

$$hf_{n+3}=\left(-\frac{17}{150}-\frac{203}{7500}u^2-\frac{68779}{15750000}u^4-\frac{1451773}{2362500000}u^6-\frac{271212551}{3307500000000}u^8\right)y_n+\left(\frac{33}{50}+\frac{107}{2500}u^2+\frac{38851}{5250000}u^4+\frac{852037}{787500000}u^6+\frac{1785369109}{12127500000000}u^8\right)y_{n+1}+\left(-\frac{93}{50}+\frac{53}{2500}u^2+\frac{11029}{5250000}u^4+\frac{147523}{787500000}u^6+\frac{4311139}{247500000000}u^8\right)y_{n+2}+\left(\frac{197}{150}-\frac{277}{7500}u^2-\frac{80861}{15750000}u^4-\frac{1546907}{2362500000}u^6-\frac{429500957}{5197500000000}u^8\right)y_{n+3}+\left(\frac{3}{25}h+\frac{16}{625}hu^2+\frac{1247}{328125}hu^4+\frac{24989}{49218750}hu^6+\frac{49915373}{757968750000}hu^8\right)f_{n+4}$$

$$y_{n+4}=\left(-\frac{3}{25}-\frac{16}{625}u^2-\frac{1247}{328125}u^4-\frac{24989}{49218750}u^6-\frac{49915373}{757968750000}u^8\right)y_n+\left(\frac{16}{25}+\frac{12}{625}u^2+\frac{593}{109375}u^4+\frac{13841}{16406250}u^6+\frac{29224187}{252656250000}u^8\right)y_{n+1}+\left(-\frac{36}{25}+\frac{48}{625}u^2+\frac{71}{15625}u^4+\frac{677}{2343750}u^6+\frac{751589}{36093750000}u^8\right)y_{n+2}+\left(\frac{48}{25}-\frac{44}{625}u^2-\frac{289}{46875}u^4-\frac{4393}{7031250}u^6-\frac{7648651}{108281250000}u^8\right)y_{n+3}+\left(\frac{12}{25}h+\frac{24}{625}hu^2+\frac{436}{109375}hu^4+\frac{3716}{8203125}hu^6+\frac{3448531}{63164062500}hu^8\right)f_{n+4}$$

It is interesting to note that as $u \to 0$, method based on polynomial basis is recovered. For TBDF, as $u \to 0$, classical Backward Differentiation Formula are recovered for $k = 2(1)4$.

3 Analysis of TBDF

Following Fatunla [20], the TBDF is represented in a block matrix form as

$$(A\otimes I)Y_{w+1}=(B\otimes I)Y_w+h(C\otimes I)F_w+h(D\otimes I)F_{w+1}, \tag{9}$$

where $Y_{w+1}=(y_{n+1},y_{n+2},...,y_{n+k})^T, Y_w=(y_{n-k+1},...,y_{n-1},y_n)^T, F_w=(f_{n+1},f_{n+2},...,f_{n+k})^T$,

$F_{w+1} = (f_{n-k+1},...,f_{n-1},f_{n+k})^T$, I is an identity matrix, $\otimes$ is the Kronecker product of matrices and A, B, C and D are $k \times k$ matrices whose entries are trigonometric coefficients and are specified as follows

$$A=\begin{bmatrix} \overline{\alpha_{1,1}} & \overline{\alpha_{2,1}} & \cdots & 0 \\ \overline{\alpha_{1,2}} & \overline{\alpha_{2,2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \cdots & 1 \end{bmatrix}, \quad B=\begin{bmatrix} 0 & 0 & \cdots & \overline{\alpha_{0,1}} \\ 0 & 0 & \cdots & \overline{\alpha_{0,2}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_0 \end{bmatrix}, \quad C=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad D=\begin{bmatrix} 1 & 0 & \cdots & \overline{\beta_{k,1}} \\ 0 & 1 & \cdots & \overline{\beta_{k,2}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_k \end{bmatrix}.$$

For $k = 2$ as an example, we have

$$A=\begin{bmatrix} -\frac{u\cos(u)+u}{\sin(u)\,(2\cos(u)+1)} & 0 \\ -\frac{2u\cos(u)+2u}{2u\cos(u)+u} & 1 \end{bmatrix}, \quad B=\begin{bmatrix} 0 & -\frac{-u\cos(u)-u}{\sin(u)\,(2\cos(u)+1)} \\ 0 & \frac{u}{2u\cos(u)+u} \end{bmatrix}, \quad C=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathrm{D}=\begin{bmatrix} 1 & -\frac{1}{2\cos(u)+1} \\ 0 & -\frac{2\sin(u)}{2u\cos(u)+u} \end{bmatrix}$$

3.1 Zero stability

Zero stability is concerned with the stability of the difference system in the limit as h tends to 0. Thus as $h \to 0 (u \to 0)$, TBDF tends to the difference system

$$BY_w - AY_{w+1} = 0, \qquad (10)$$

where A and B are $k \times k$ constant matrices.

TBDF is then zero stable if the roots $\tau_j, j = 1,2,...,k$ of the first characteristic polynomial $\sigma(\tau)$ specified by $\sigma(\tau) = \det[B - \tau A] = 0$ and satisfies $|\tau_j| \le 1$ and for those roots with $|\tau_j| = 1$ are simple [20]. From our calculations, for each member of TBDF (for different k),

$$\sigma(\tau) = \tau^k (k-1) \Rightarrow \tau_j = 0 (j = 1,2,...k-1) \cdot \qquad (11)$$

Hence, our methods are zero stable.

3.2 Local Truncation Error (LTE) and order of TBDF

Since the TBDF is made up of generalized linear multistep methods with trigonometric coefficients, we associate the TBDF with a linear operator defined by

$$L_w = [y(x_n);h] = y(x_n + kh) - \left(h\beta_k(u) y'(x_n + kh) + \sum_{j=0}^{k-1} \alpha_j(u) y(x_n + jh) \right) \cdot \qquad (12)$$

Assuming that $y(x_n)$ is sufficiently differentiable, expanding with Taylor series expansions of $y(x_n + kh), y(x_n + jh)$ and $y'(x_n + kh)$ about the point x_n and substituting the coefficients $\alpha_j(u)$ and $\beta_k(u)$ into equations (12), we obtain the algebraic order, the Local Truncation Error as presented below in Table 1.

Table 1. Local truncation error and order of TBDF

k	LTE	Order (p)
2	$\begin{bmatrix} -\frac{5h^3}{18}\left(D(y)(x)\omega^2 + D^{(3)}(y)(x)\right) \\ -\frac{2h^3}{9}\left(D(y)(x)\omega^2 + D^{(3)}(y)(x)\right) \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$
3	$\begin{bmatrix} \frac{7h^4}{66}\left(D^{(2)}(y)(x)\omega^2 + D^{(4)}(y)(x)\right) \\ -\frac{17h^3}{132}\left(D^{(2)}(y)(x)\omega^2 + D^{(4)}(y)(x)\right) \\ -\frac{3h^3}{22}\left(D^{(2)}(y)(x)\omega^2 + D^{(4)}(y)(x)\right) \end{bmatrix}$	$\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$
4	$\begin{bmatrix} -\frac{29h^5}{500}\left(\omega^2 D^{(3)}(y)(x) + D^{(5)}(y)(x)\right) \\ \frac{31h^5}{750}\left(\omega^2 D^{(3)}(y)(x) + D^{(5)}(y)(x)\right) \\ -\frac{37h^5}{500}\left(\omega^2 D^{(3)}(y)(x) + D^{(5)}(y)(x)\right) \\ -\frac{12h^5}{125}\left(\omega^2 D^{(3)}(y)(x) + D^{(5)}(y)(x)\right) \end{bmatrix}$	$\begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$

Table 1 shows that the TBDF has at least order $p \geq 1$. Therefore, we remark that the TBDF (for each k) is consistent.

3.3 Convergence of TBDF

The necessary and sufficient condition for a method to be convergent is that it must be zero stable and consistent [21,22]. Since TBDF (for each k) is both zero stable and consistent, we, therefore, remark that it is convergent.

3.4 Linear stability and region of absolute stability of TBDF

Applying the block method to the test equations $y' = \lambda y$ and $y'' = \lambda^2 y$ and letting $z = \lambda h$, $u = \omega h$ yields $Y_{w+1} = \sigma(z,u) Y_w$, where. The rational function $\sigma(z,u)$ is called the stability function which is used to determine the stability region of the $k-$step TBDF.

***Definition 3.1* [23].** A region of stability is a region in the $z-u$ plane throughout which the spectral radius $\left|\rho\left(\sigma(z,u)\right)\right| \leq 1$ The $z-u$ stability constructed for TBDF are plotted as presented in Figs. 1-3 respectively.

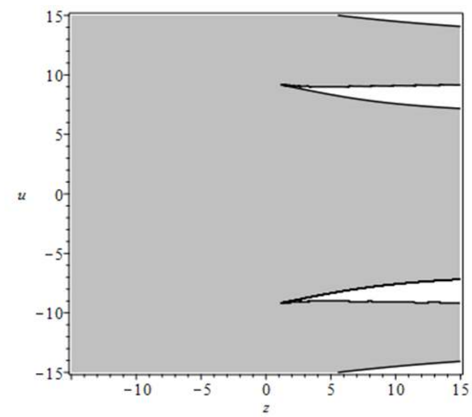

Fig. 1. The z - u plot for TBDF1

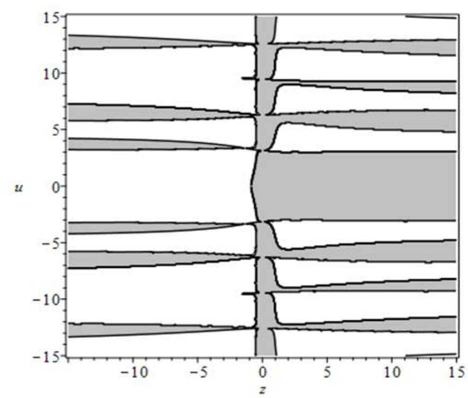

Fig. 2. The z - u plot for TBDF2

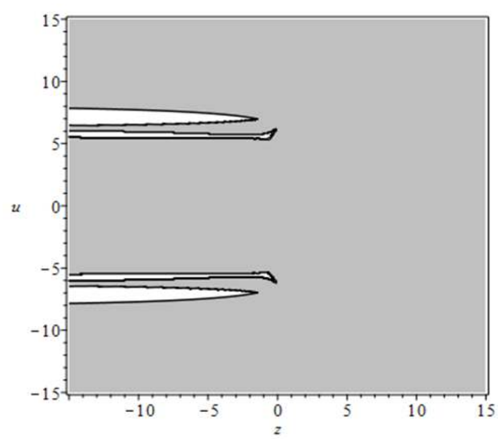

Fig. 3. The $z - u$ plot for TBDF3

4 Numerical Examples

This section discusses the performance, efficiency, and accuracy of the fourth order TBDF on a variety of well-known oscillatory IVPs. The conditions for the numerical comparison with results from existing methods in the literature include the accuracy which is measured by the maximum global errors and the computational efficiency measured in terms of the number of function evaluation (NFE) required by the method. We noted that the methods developed in this paper could be implemented for all values of N. However, for the purpose of comparison the N values used in the existing literature were used therein.

Problem 1: Inhomogeneous stiff oscillatory problem

Consider the inhomogeneous stiff oscillatory problem given $y' = -\lambda(y - \sin x) + \cos x,\ y(0) = 0$ with exact solution given as $y(x) = \sin x$. This problem was solved for $\lambda = 10^{-6}$ by an order 8 Absolute stable Runge-Kutta Collocation method (ARKC) in Vigo-Aguiar and Ramos 2007 for $h = \frac{1}{2^n}, n = -1(1)2$ in the interval $[0,10]$. The computed and exact solution correct to 20 significant figure is presented in Table 2 while the numerical comparison of the maximum absolute error of TBDF with ARKC is given in Table 3.

Table 2. Computed and exact solutions of problem 1

	TBDF	
n	**Computed solution**	**Exact solution**
- 1	-0.27941549702122940635	-0.27941549819892587281
0	0.65698659871878299402	0.65698659871878909040
1	-0.35078322768961984811	-0.35078322768961984812
2	-0.31951919362227363990	-0.31951919362227363990

Table 3. Comparison of maximum errors for problem 1

n	*TBDF*		*ARKC*	
	NFE	*Err*	*NFE*	*Err*
- 1	*6*	1.18×10^{-09}	*35*	1.97×10^{-11}
0	*11*	6.09×10^{-15}	*70*	2.75×10^{-13}
1	*21*	6.67×10^{-21}	*140*	5.59×10^{-15}
2	*41*	6.54×10^{-27}	*280*	3.33×10^{-16}

It is evident in Table 2 and Figure 4 that TBDF having small error, and fewer number of function evaluation, is more accurate and efficient than ARKC of order 8.

Problem 2:

Consider the following first order oscillatory IVP $y' = y\cos x, y(0) = 1$ whose exact solution is given as $y(x) = \exp(\sin x)$. The computed and exact solution correct to 20 significant figure are presented in Table 4 while the numerical results of TBDF with $\omega = 0.5$ in comparison with the Fifth order Exponentially-Fitted Runge Kutta (EFRK) with six stages in Vanden Berghe et al. are presented in Table 5.

Table 4. Computed and exact solutions of problem 2

	TBDF	
N	**Computed solution**	**Exact solution**
230	1.69593806157911153273	1.69593806146894311240
430	1.69826814056969219873	1.69826814056071872950
800	1.70815837420424001101	1.70815837420349300866

Table 5. Comparison of maximum errors for problem 2

TBDF			**EFRK**		
N	*NFE*	*Err*	**N(Rejected)**	*NFE*	*Err*
230	*231*	1.10×10^{-10}	23(5)	*430*	6.88×10^{-06}
430	*431*	8.97×10^{-12}	43(8)	*810*	4.51×10^{-08}
800	*801*	7.47×10^{-13}	80(9)	*1513*	3.13×10^{-09}

It is clear from Table 3 and Fig. 5 that TBDF a fourth order method shows superiority in terms of accuracy and efficiency over a fifth order EFRK in the literature.

Problem 3: Oscillatory Stiff Problem (Sofoluwe et al. [7])

As our third test, the stiff oscillatory problem $y' = -100(y - \sin x), y(0) = 0$ is considered for step-length $\frac{\pi}{60}$ in the interval $0 \le x \le 2x$ and $\omega = 1$ with the analytical solution of $y(x) = \frac{\sin x - 0.01\cos x + \exp(-100x)}{1.0001}$ The computed and exact solution correct to 20 significant figure is presented in Table 6 while the numerical results are compared with the Block Backward Differentiation Formula of order 4 (BBDF1) and Block Backward Differentiation Formula of order 5 (BBDF2) of Sofoluwe et al. [7] as presented below in Tables 7.

Table 6. Computed and exact solutions of problem 3

	TBDF	
x	**Computed solution**	**Exact solution**
$\frac{\pi}{6}$	0.39756380133844526556	0.39756380133844526556
$\frac{\pi}{2}$	0.99337727300829392581	0.99337727300829597261
π	0.00999900009999000100	0.00999900009999000100
$\frac{3\pi}{2}$	-0.99337727300829597261	-0.99337727300829597261
2π	-0.00999900009999000100	-0.00999900009999000100

Table 7. Comparison of maximum errors with $h = \frac{\pi}{60}$ for Problem 3

x	TBDF			BBDF1			BBDF2		
	N	**Err**	**NFE**	**N**	**Err**	**NFE**	**N**	**Err**	**NFE**
$\frac{\pi}{6}$	10	2.37E-06	11	10	1.64E-04	11	10	3.69E-03	11
$\frac{\pi}{2}$	30	2.05E-15	31	30	1.67E-04	31	30	2.37E-04	31
π	60	6.95E-30	61	60	3.81E-07	61	60	6.70E-04	61
$\frac{3\pi}{2}$	90	4.36E-30	91	90	9.05E-05	91	90	2.22E-04	91
2π	120	6.50E-31	121	120	4.08E-05	121	120	6.70E-04	121

Although in Table 5 TBDF required the same number of function evaluations with the BBDF1 and BBDF2 respectively, it is more accurate and efficient than the two methods as shown in Fig. 6. In fact, the accuracy of TBDF is at least twice that of the other two methods.

Problem 4: The Cosine Problem

Consider the IVP given by $y' = -2\pi \sin 2\pi x - \frac{1}{\in}(y - \cos 2\pi x),\ y(0) = 1$ whose solution in closed form is given as $y(x) = \cos 2\pi x$. It should be noted that as $\in \to 0$ the equation becomes increasingly stiff. Also, the term $-2\pi \sin(2\pi x)$ in the equation is treated explicitly and the term $\frac{-(y - \cos 2\pi x)}{\in}$ implicitly. This problem was considered in the interval $0 \le x \le 10$ for $\in = 10^{-3}$ by Suleiman et al. [8] using New Variable Step size Block Backward Differentiation Formula (NVSBBDF) method of order 4 while Musa et al. [8] used Variable Step size Superclass Block Backward Differentiation Formula (VSSBBDF) method of order 5. The results of TBDF for $h = 2^{-i-1}, i = 1, 2, 3$ are displayed in Table 8 and the comparison with NVSBBDF and VSSBBDF as shown in Table 9.

Table 8. Computed and exact solutions of problem 4

	TBDF	
N	**Computed solution**	**Exact solution**
40	-0.00000006527353596669	0.00000000000000000000
80	-0.70710678118622676515	-0.70710678118654752440
160	0.38268343236508977198	0.38268343236508977173

Table 9. Comparison of maximum errors for problem 4

TBDF		NVSBBDF		VSBBDF		NBDF	
N	**Err**	**N (Rejected)**	**Err**	**N (Rejected)**	**Err**	**N (Rejected)**	**Err**
40	6.53E-08	66 (1)	5.16E-05	76 (4)	5.25E-05	164 (38)	8.47E-05
80	3.21E-13	116 (0)	1.54E-06	166 (7)	2.12E-06	319 (70)	3.43E-06
160	2.47E-19	402 (4)	3.12E-08	476 (19)	4.32E-08	574 (100)	1.33E-07

The results in Table 5 show the superiority of TBDF which is applied with fixed step length in terms of accuracy over the NVSBBDF, VSBBDF, and NBDF that are applied in variable step length respectively.

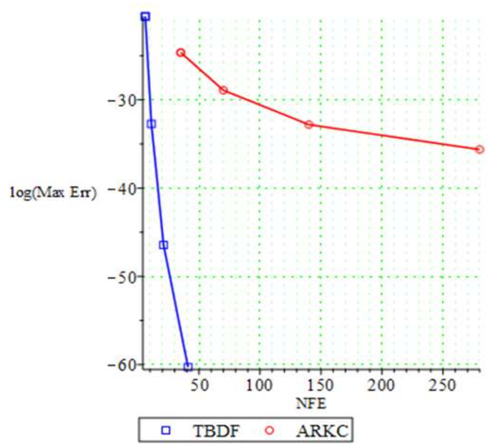

Fig. 4. Efficiency curve for problem 1

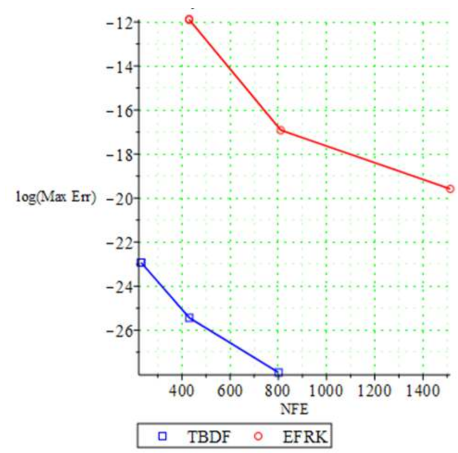

Fig. 5. Efficiency curve for problem 2

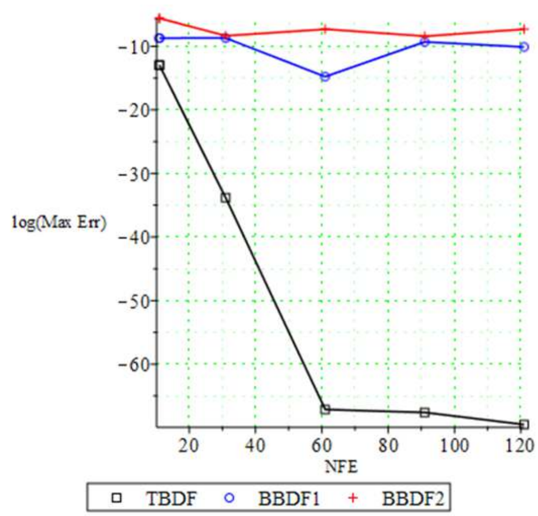

Fig. 6. Efficiency curve for problem 3

5 Conclusion

A family of block backward differentiation methods with trigonometric coefficients is presented in this paper for the integration of first-order periodic IVPs. The methods are obtained using multistep collocation technique. The stability properties of each member of the family were analyzed. The numerical examples considered showed that the family of TBDF is an accurate and efficient integrator for first-order periodic initial value problems.

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Competing Interests

Author has declared that no competing interests exist.

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