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DIRECTED PATHOS TOTAL DIGRAPH OF AN ARBORESCENCE

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ABSTRACT. For an arborescence A_r, a *directed pathos total digraph* $Q = DPT(A_r)$ has vertex set $V(Q) = V(A_r) \cup A(A_r) \cup P(A_r)$, where $V(A_r)$ is the vertex set, $A(A_r)$ is the arc set, and $P(A_r)$ is a directed pathos set of A_r. The arc set $A(Q)$ consists of the following arcs: ab such that $a, b \in A(A_r)$ and the head of a coincides with the tail of b; uv such that $u, v \in V(A_r)$ and u is adjacent to v; au (ua) such that $a \in A(A_r)$ and $u \in V(A_r)$ and the head (tail) of a is u; Pa such that $a \in A(A_r)$ and $P \in P(A_r)$ and the arc a lies on the directed path P; P_iP_j such that $P_i, P_j \in P(A_r)$ and it is possible to reach the head of P_j from the tail of P_i through a common vertex, but it is possible to reach the head of P_i from the tail of P_j. For this class of digraphs we discuss the planarity; outerplanarity; maximal outerplanarity; minimally nonouterplanarity; and crossing number one properties of these digraphs. The problem of reconstructing an arborescence from its directed pathos total digraph is also presented.

Index Terms: Line digraph; directed path number; crossing number; inner vertex number.

1. Introduction

Notations and definitions not introduced here can be found in [1]. There are many graph valued functions (or graph operators) with which one can construct a new graph from a given graph, such as the line graphs, the total graphs, and their generalizations. The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have a common vertex. This concept was originated

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with Whitney [2]. Harary and Norman [3] extended the concept of line graph of a graph and introduced the concept of line digraph of a directed graph. The *line digraph* $L(D)$ of a digraph D has the arcs of D as vertices. There is an arc from D-arc pq towards D-arc uv if and only if $q = u$.

Behzad [4] introduced the concept of total graph of a graph. The *total graph* of a graph G, written $T(G)$, is the graph whose vertices can be put in one-to-one correspondence with the vertices and edges of G in such a way that two vertices of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent, where the vertices and edges of G are called its *members*. Gary Chatrand and James Stewart [5] extended the concept of total graph of a graph to the directed case there by introducing the total digraph.

The *total digraph* of a directed graph D, written $T(D)$, is the digraph whose vertices are in one-to-one correspondence with the vertices and arcs of D and such that the vertex u is adjacent to the vertex v in $T(D)$ if and only if the element corresponding to u is adjacent to the element corresponding to v in D.

The concept of *pathos* of a graph G was introduced by Harary [6] as a collection of minimum number of edge disjoint open paths whose union is G. The *path number* of a graph G is the number of paths in any pathos. The *path number* of a tree T equals k, where $2k$ is the number of odd degree vertices of T. Stanton and Cowan [7] calculated the path number of certain classes of graphs like trees and complete graphs. Gudagudi [8] extended the concept of pathos of graphs to trees there by introducing the concept called *pathos line graph* of a tree. A *pathos line graph* of a tree T, written $PL(T)$, is a graph whose vertices are the edges and paths of a pathos of T, with two vertices of $PL(T)$ adjacent whenever the corresponding edges of T are adjacent or the edge lies on the corresponding path of the pathos.

Since the pattern of pathos for a tree is not unique, the corresponding pathos line graph is also not unique. See Figure 1 for an example of a tree and its pathos line graph.

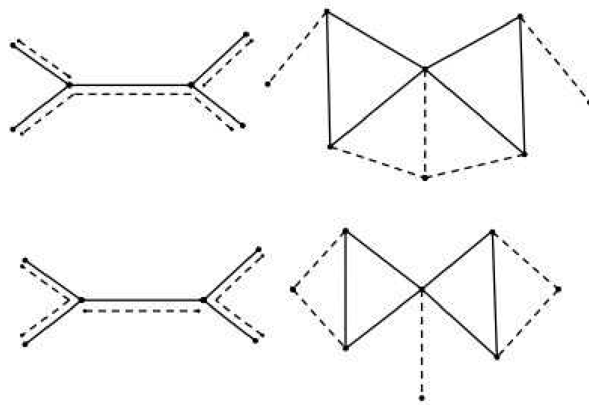

Figure 1.

It is the object of this paper to extend the concept of pathos of a tree to the directed case by introducing the concept called *directed pathos total digraph* of an arborescence and to develop some of its properties.

2. Preliminaries

We need some concepts and notations on graphs and directed graphs. A *graph* $G = (V, E)$ is a pair, consisting of some set V, the so-called *vertex set*, and some subset E of the set of all 2-element subsets of V, the *edge set*. If a path starts at one vertex and ends at a different vertex, then it is called an *open path*.

A graph G is *planar* if it has a drawing without crossings. For a planar graph G, the *inner vertex number* $i(G)$ is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. If a planar graph G is embeddable in the plane so that all the vertices are on the boundary of the exterior region, then G is said to be *outerplanar*, i.e., $i(G) = 0$. An outerplanar graph G is *maximal outerplanar* if no edge can be added without losing outerplanarity. A graph G is said to be *minimally nonouterplanar* if $i(G)$=1. The least number of edge crossings of a graph G, among all planar embeddings of G, is called the *crossing number* of G and is denoted by $\mathrm{cr}(G)$.

A *directed graph* (or just *digraph*) D consists of a finite non-empty set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of ordered pairs of distinct vertices called *arcs*. Here $V(D)$ is the *vertex set* and $A(D)$ is the *arc set* of D. For an arc (u, v) or uv in D, the first vertex u is its *tail* and the second vertex v is its *head*. The head and tail of an arc are its *end-vertices*. For an arc $e = (u, v)$, we say that u is a *neighbor* of v; and u is *adjacent* to e and e is *adjacent* to v. A vertex u is *adjacent* to v if the arc uv is in D; u is *adjacent* from v if vu is in D. A digraph without any arcs is said to be *totally disconnected*. For a digraph $D = (V, A)$, the *out-neighbourhood* $N^+(v)$ of a vertex v is the set of all

vertices w with $vw \in A$. The *in-neighbourhood* $N^-(v)$ of a vertex v is the set of all vertices w with $wv \in A$.
The *out-degree* $d^+(v)$ or *in-degree* $d^-(v)$ of a vertex v is the cardinality of the out-neighbourhood or in-neighbourhood of v, respectively. The *total degree* $td(v)$ of a vertex v is the number of arcs incident with v, that is, $td(v) = d^-(v)+d^+(v)$. A *source* is any vertex of in-degree zero and a *sink* is a vertex of out-degree zero. A vertex is *isolated* if both out-degree and in-degree are zero.
A *semi-directed path joining* v_1 *and* v_n is a collection of distinct vertices $v_1, v_2, \ldots, v_n$ together with $n-1$ vertices, one from each pair of arcs, v_1v_2 or v_2v_1; v_2v_3 or v_3v_2, $\ldots$, $v_{n-1}v_n$ or v_nv_{n-1}. A *semi-directed cycle* is obtained from a semi directed path on adding an arc joining the terminal vertex and the initial vertex of the semi-directed path.
A digraph is *strongly connected* (or just *strong*) if every two vertices are mutually reachable. A digraph is *unilaterally* connected or *unilateral* if for any two vertices, at least one is reachable from the other; it is *strictly unilateral* if it is unilateral but not strong.
A digraph is *weakly connected* or *weak* if every two vertices are joined by a semi-directed path; it is *strictly weak* if it is weak but not unilateral. A *block* B of a digraph D is a maximal weak subgraph of D, which has no cut-vertex v such that $B-v$ is disconnected. An entire digraph is a block if it has only one block. There are exactly three categories of blocks: strong, strictly unilateral, and strictly weak.
Digraphs that can be drawn without crossings between arcs (except at end vertices) are called *planar digraphs*. Clearly this property does not depend on the orientation of the arcs and hence we ignore the orientation while defining the planarity; outerplanarity; maximal outerplanarity; and minimally nonouterplanarity of a digraph. Furthermore, since most of the results and definitions of undirected graphs are valid for planar digraphs as far as their underlying graphs are concerned, the following definitions hold good for planar digraphs. A digraph D is said to be *outerplanar* if $i(D) = 0$ and *minimally nonouterplanar* if $i(D) = 1$.
The following result characterizes maximal outerplanar graphs, and the same can be used to check the maximal outerplanar property of a digraph.

Theorem 2.1. [9] *Every maximal outerplanar graph G with n vertices has $2n-3$ edges.*

3. Definition of $DPT(A_r)$

Definition 3.1. An *arborescence* is a directed graph in which, for a vertex u called the *root* and any other vertex v, there is exactly one directed path from u to v.

We shall use A_r to denote an arborescence.

Definition 3.2. A *root arc* of an arborescence A_r is an arc which is directed out of the root of A_r, i.e., a *root arc* of A_r an arc whose tail is the root of A_r.

Definition 3.3. If a directed path $\vec{P}_n$ of order n $(n \geq 2)$ starts at one vertex and ends at a different vertex, then $\vec{P}_n$ is called an *open directed path.*

Definition 3.4. The *directed pathos* of an arborescence A_r is defined as a collection of minimum number of arc disjoint open directed paths whose union is A_r.

Definition 3.5. The *directed path number* $k^{'}$ of an arborescence A_r is the number of directed paths in any directed pathos of A_r, and is equal to the number of sinks in A_r, i.e., $k^{'}$ = number of sinks in A_r.

Note that the directed path number $k^{'}$ of an arborescence A_r is minimum only when the out-degree of the root of A_r is exactly one. Therefore, unless otherwise specified, the out-degree of the root of every arborescence is exactly one. Finally, we assume that the direction of the directed pathos is along the direction of the arcs in A_r.

Definition 3.6. For an arborescence A_r, a *directed pathos total digraph* $Q = DPT(A_r)$ has vertex set $V(Q) = V(A_r) \cup A(A_r) \cup P(A_r)$, where $V(A_r)$ is the vertex set, $A(A_r)$ is the arc set, and $P(A_r)$ is a directed pathos set of A_r. The arc set $A(Q)$ consists of the following arcs: ab such that $a, b \in A(A_r)$ and the head of a coincides with the tail of b; uv such that $u, v \in V(A_r)$ and u is adjacent to v; au (ua) such that $a \in A(A_r)$ and $u \in V(A_r)$ and the head (tail) of a is u; Pa such that $a \in A(A_r)$ and $P \in P(A_r)$ and the arc a lies on the directed path P; P_iP_j such that $P_i, P_j \in P(A_r)$ and it is possible to reach the head of P_j from the tail of P_i through a common vertex, but it is possible to reach the head of P_i from the tail of P_j.

Since the pattern of directed pathos for an arborescence is not unique, the corresponding directed pathos total digraph is also not unique. But it is cleared from the definition of the directed path number $k^{'}$ and $DPT(A_r)$ that, for a directed path $\vec{P}_n$ of order n $(n \geq 2)$, the corresponding directed pathos total digraph is unique. Furthermore, one can observe easily that, for different pattern of directed pathos of an arborescence whose underlying graph is a star graph $K_{1,n}$ on $n \geq 3$ vertices, the corresponding directed pathos total digraphs are isomorphic. A digraph $A_r^{'}$ is a directed pathos total digraph if there exists an arborescence A_r such that $A_r^{'} = DPT(A_r)$. See Figure 2 for an example of an arborescence and its directed pathos total digraph.

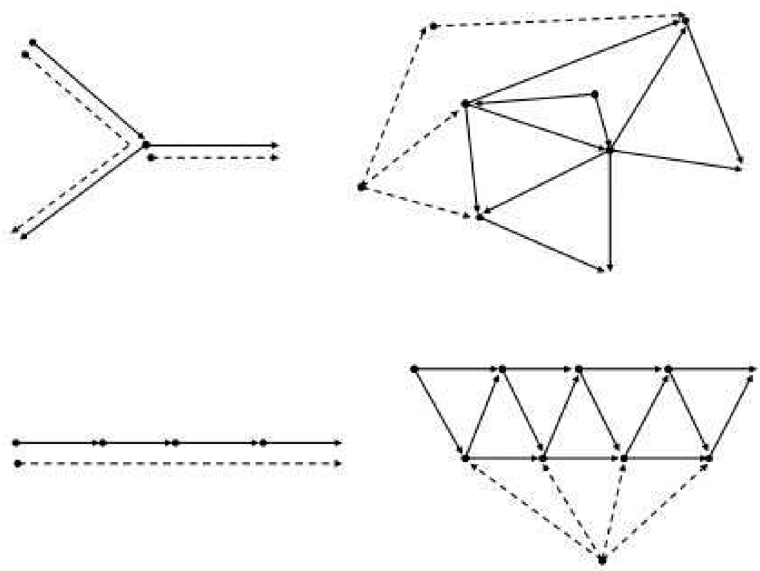

Figure 2.

4. A criterion for directed pathos total digraphs

The main objective is to determine a necessary and sufficient condition that a digraph be a directed pathos total digraph.

A *complete bipartite digraph* is a directed graph D whose vertices can be partitioned into non-empty disjoint sets A and B such that each vertex of A has exactly one arc directed towards each vertex of B and such that D contains no other arc.

Let A_r be an arborescence with vertex set $V(A_r) = \{v_1, v_2, \ldots, v_n\}$ and a directed pathos set $P(A_r) = \{P_1, P_2, \ldots, P_t\}$. We consider the following cases.

Case 1. Let v be a vertex of A_r with $d^-(v) = \alpha$ and $d^+(v) = \beta$. Then α arcs coming into v and the β arcs going out of v give rise to a complete bipartite subdigraph with α tails and β heads and $\alpha \cdot \beta$ arcs joining each tail with each head. This is the decomposition of $L(A_r)$ (i.e., the line digraph of A_r) into mutually arc disjoint complete bipartite subdigraphs.

Case 2. An arc $e = (u, v)$ with $d^+(u) = d^-(v) = 1$ give rise to a complete bipartite subdigraph with u as the tail and v head. This contributes $n - 1$ arcs to $DPT(A_r)$.

Case 3. An arc $e = (u, v)$ with $d^+(u) = d^-(v) = 1$ give rise to a complete bipartite subdigraph with u as the tail and e head. This contributes $n - 1$ arcs to $DPT(A_r)$.

Case 4. An arc $e = (u, v)$ with $d^+(u) = d^-(v) = 1$ give rise to a complete bipartite subdigraph with e as the tail and v head. This also contributes $n-1$ arcs to $DPT(A_r)$.
Case 5. Let P_j be a directed path which lies on $\alpha^{'}$ arcs in A_r. Then $\alpha^{'}$ arcs give rise to a complete bipartite subdigraph with a single tail P_j and $\alpha^{'}$ heads and $\alpha^{'}$ arcs joining P_j with each head. This again contributes $n-1$ arcs to $DPT(A_r)$.
Case 6. Let P_j be a directed path and let $\beta^{'}$ be the number of directed paths whose head is reachable from the tail of P_j through a common vertex in A_r. Then $\beta^{'}$ arcs give rise to a complete bipartite subdigraph with a single tail P_j and $\beta^{'}$ heads and $\beta^{'}$ arcs joining P_j with each head. This contributes $k^{'}-1$ arcs to $DPT(A_r)$.
Hence by all the cases above, $Q = DPT(A_r)$ is decomposed into mutually arc-disjoint complete bipartite subdigraphs with $V(Q) = V(A_r) \cup A(A_r) \cup P(A_r)$ and arc sets, (i) $\cup_{i=1}^{n} X_i \times Y_i$, where X_i and Y_i are the sets of in-coming and out-going arcs at v_i of A_r, respectively; (ii) four times the size of A_r, i.e., $4(n-1)$; and (iii) $k^{'}-1$.
Conversely, let $A_r^{'}$ be a digraph of the type described above. Let $t_1, t_2, \ldots, t_l$ be the vertices corresponding to complete bipartite subdigraphs $T_1, T_2, \ldots, T_l$ of Case 1, respectively; and let $w^1, w^2, \ldots, w^t$ be the vertices corresponding to complete bipartite subdigraphs $P_1^{'}, P_2^{'}, \ldots, P_t^{'}$ of Case 5, respectively. Finally, let t_0 be a vertex chosen arbitrarily.
For each vertex v of the complete bipartite subdigraphs $T_1, T_2, \ldots, T_l$, we draw an arc a_v as follows:

- If $d^+(v) = 1$, $d^-(v) = 0$, then $a_v := (t_0, t_i)$, where i is the base (or index) of T_i such that $v \in Y_i$.
- If $d^+(v) > 0$, $d^-(v) > 0$, then $a_v := (t_i, t_j)$, where i and j are the indices of T_i and T_j such that $v \in X_j \cap Y_i$.
- If $d^+(v) = 0$, $d^-(v) = 1$, then $a_v := (t_j, w^n)$ for $1 \leq n \leq t$, where j is the base of T_j such that $v \in X_j$.

Note that, in (t_j, w^n) no matter what the value of j is, n varies from 1 to t such that the number of arcs of the form (t_j, w^n) is exactly t.
We now mark the directed pathos as follows. It is easy to observe that the directed path number $k^{'}$ equals the number of subdigraphs of Case 5. Let $\psi_1, \psi_2, \ldots, \psi_t$ be the number of heads of subdigraphs $P_1^{'}, P_2^{'}, \ldots, P_t^{'}$, respectively. Suppose we mark the directed path P_1. For this we choose any ψ_1 number of arcs and mark P_1 on ψ_1 arcs. Similarly, we choose ψ_2 number of arcs and mark P_2 on ψ_2 arcs. This process is repeated until all the directed paths of a directed pathos are marked. The digraph A_r with directed pathos thus constructed apparently has $A_r^{'}$ as directed pathos total digraph. Thus we have,

Theorem 4.1. *A digraph $A_r^{'}$ is a directed pathos total digraph of an arborescence A_r if and only if $V(A_r^{'}) = V(A_r) \cup A(A_r) \cup P(A_r)$ and arc sets, (i) $\cup_{i=1}^{n} X_i \times$*

Y_i, where X_i and Y_i are the sets of in-coming and out-going arcs at v_i of A_r, respectively; (ii) four times the size of A_r, i.e., $4(n-1)$; and (iii) $k^{'}-1$.

Given a directed pathos total digraph Q, the proof of the sufficiency of Theorem above shows how to find an arborescence A_r such that $DPT(A_r) = Q$. This obviously raises the question of whether Q determines A_r uniquely. Although the answer to this in general is no, the extent to which A_r is determined is given as follows.

One can check easily that using reconstruction procedure of the sufficiency of Theorem above, any arborescence (without directed pathos) is uniquely reconstructed from its directed pathos total digraph. Since the pattern of directed pathos for an arborescence is not unique, there is freedom in marking directed pathos for an arborescence in different ways. This clearly shows that if the directed path number is one, any arborescence with directed pathos is uniquely reconstructed from its directed pathos total digraph. It is known that a directed path is a special case of an arborescence. Since the directed path number $k^{'}$ of a directed path $\vec{P_n}$ of order n ($n \geq 2$) is exactly one, a directed path with directed pathos is uniquely reconstructed from its directed pathos total digraph.

5. Properties of $DPT(A_r)$

In this section we present some of the properties of $DPT(A_r)$.

Property 5.1. *For an arborescence A_r, $L(A_r) \subseteq T(A_r) \subseteq DPT(A_r)$, where $\subseteq$ is the subdigraph notation.*

Property 5.2. *If the in-degree (out-degree) of a vertex v in A_r is n, then the in-degree (out-degree) of the corresponding vertex v in $DPT(A_r)$ is $2n$.*

Property 5.3. *The in-degree of the vertex v in $DPT(A_r)$ corresponding to the root arc of A_r is two.*

Property 5.4. *The in-degree of the vertex v in $DPT(A_r)$ corresponding to a pendant arc of A_r is two.*

Property 5.5. *A directed pathos total digraph $DPT(A_r)$ of an arborescence A_r does not contain any vertex v such that $DPT(A_r)$ is disconnected. Hence $DPT(A_r)$ is a block.*

Property 5.6. *Every pair of vertices and arcs of $DPT(A_r)$ lie on a semi-directed cycle.*

Property 5.7. *For any three distinct vertices u, v, and w, there is a semi-directed path joining u and w which contains v.*

Property 5.8. *For any three distinct vertices u, v, and w, there is a semi-directed path joining u and w which does not contains v.*

Property 5.9. *Every $DPT(A_r)$ is either strictly unilateral or strictly weak.*

In order to prove the next property, we need the following Theorem and definitions.

Theorem 5.10. [10] *Let D be an acyclic digraph with precisely one source x in D. Then for every $v \in V(D)$, there is an $(x, v)-$directed path in D.*

Definition 5.11. A *transmitter* is a vertex v whose out-degree is positive and whose in-degree is zero, i.e., $d^+(v) > 0$ and $d^-(v) = 0$.

Definition 5.12. A *carrier* is a vertex v whose out-degree and in-degree are both one, i.e., $d^+(v) = d^-(v) = 1$.

Definition 5.13. A *receiver* is a vertex v whose out-degree is zero and whose in-degree is positive, i.e., $d^+(v) = 0$ and $d^-(v) > 0$.

Definition 5.14. A vertex v is said to be *ordinary* if $d^+(v) > 0$ and $d^-(v) > 0$.

Definition 5.15. A *directed pathos vertex* is a vertex corresponding to the directed path of a directed pathos of A_r.

Proposition 5.16. *Let A_r be an arborescence of order n $(n \geq 2)$ with v_1 and $e_1 = (v_1, v_2)$ as the root and root arc of A_r, respectively. Then there exists exactly one vertex v with $d^+(v) > 0$ and $d^-(v) = 0$ (i.e., transmitter), and for every vertex $w \in DPT(A_r)$ (except for the vertex v_1), there is an $(v, w)-$ directed path in $DPT(A_r)$.*

Proof. Let A_r be an arborescence with vertex set $V(A_r) = \{v_1, v_1, \ldots, v_n\}$ and arc set $A(A_r) = \{e_1, e_2, \ldots, e_{n-1}\}$ such that v_1 and $e_1 = (v_1, v_2)$ are the root and root arc of A_r, respectively. Then the vertices $e_2, e_3, \ldots, e_{n-1}$ are reachable from e_1 by a unique directed path in $L(A_r)$. Let $P(A_r) = \{P_1, P_2, \ldots, P_{k'}\}$ be a directed pathos set of A_r such that P_1 lies on the arc e_1. Since the direction of the directed pathos is along the direction of the arcs in A_r, $d^+(v_1) = 2, d^-(v_1) = 0$; $d^+(P_1) > 0$, $d^-(P_1) = 0$; and the remaining vertices are either receiver or carrier or ordinary, in $DPT(A_r)$. Clearly, $DPT(A_r)$ is acyclic. By Theorem 5.10, for every (except v_1) vertex $w \in DPT(A_r)$, there is an $(P_1, w)-$ directed path in $DPT(A_r)$. This completes the proof. □

When defining any class of digraphs, it is desirable to know the order and size of each; it is easy to determine for $DPT(A_r)$.

Proposition 5.17. *Let A_r be an arborescence with n vertices $v_1, v_2, \ldots, v_n$ and k' sinks. Then the order and size of $DPT(A_r)$ are $2n+k'-1$ and $4n+\sum_{i=1}^{n} d^-(v_i)\cdot d^+(v_i) + k' - 5$, respectively.*

Proof. If A_r has n vertices and k' sinks, then it follows immediately that $DPT(A_r)$ contains $n + n - 1 + k' = 2n + k' - 1$ vertices. Furthermore, every arc of $DPT(A_r)$ corresponds to an arc in A_r (there are $n - 1$ arcs); adjacent arcs in A_r (this is given by $\sum_{i=1}^{n} d^{-}(v_i)\cdot d^{+}(v_i)$); an arc adjacent to a vertex in A_r (there are $n-1$ of these); a vertex adjacent to an arc in A_r (there are $n-1$ of these); the arcs lie on the directed paths of a directed pathos of A_r (there are also $n-1$ of these); and the arcs whose end-vertices are the directed pathos vertices (this is given by $k^{'}-1$). Therefore, $DPT(A_r)$ has

$$(n-1)+\sum_{i=1}^{n} d^{-}(v_i)\cdot d^{+}(v_i)+3(n-1)+k^{'}-1 = 4n+\sum_{i=1}^{n} d^{-}(v_i)\cdot d^{+}(v_i)+k^{'}-5$$

arcs. □

6. Characterization of $DPT(A_r)$

6.1. Planar directed pathos total digraphs. We now characterize the digraphs whose $DPT(A_r)$ is planar.

Theorem 6.1. *A directed pathos total digraph $DPT(A_r)$ of an arborescence A_r is planar if and only if the underlying graph of A_r is a star graph $K_{1,n}$ on $n \leq 3$ vertices.*

Proof. Suppose $DPT(A_r)$ is planar. Assume that the underlying graph of A_r is a star graph $K_{1,n}$ on $n \geq 4$ vertices. Suppose that $A_r = K_{1,4}$. Let $V(A_r) = \{v_1, v_2, v_3, v_4, v_5\}$ be the vertex set and $A(A_r) = \{e_1, e_2, e_3, e_4\}$ be the arc set of A_r such that v_1 and $e_1 = (v_1, v_2)$ are the root and root arc of A_r, respectively; and $e_i = (v_2, v_{i+1})$ for $2 \leq i \leq 4$. Then (e_1, e_{i+1}) for $1 \leq i \leq 3$; (v_1, v_2); (v_2, v_{i+1}) for $2 \leq i \leq 4$; (e_i, v_{i+1}) for $1 \leq i \leq 4$; (v_1, e_1); and (v_2, e_i) for $2 \leq i \leq 4$ are the arcs of $T(A_r)$. Let $P(A_r) = \{P_1, P_2, P_3\}$ be a directed pathos set of A_r such that P_1 lies on the arcs $(v_1, v_2), (v_2, v_3)$; P_2 lies on (v_2, v_4); and P_3 lies on (v_2, v_5). Then the directed pathos vertex P_1 is a neighbor of the vertices v_1v_2, v_2v_3, P_2, P_3; P_2 is a neighbor of v_2v_4; and P_3 is a neighbor of v_2v_5. This shows that the crossing number of $DPT(A_r)$ is one, i.e., $\mathrm{cr}(DPT(A_r)) = 1$, a contradiction (see Figure 3).

Conversely, suppose that the underlying graph of A_r is a star graph $K_{1,n}$ on $n \leq 3$ vertices. We consider the following three cases.

Case 1. Suppose that the underlying graph of A_r is $K_{1,1}$, i.e., $\vec{P_2}$. Then the underlying graph of $DPT(A_r)$ is $K_{1,3}+e$, i.e., the kite graph. Clearly $DPT(A_r)$ is planar.

Case 2. Suppose that the underlying graph of A_r is $K_{1,2}$, i.e., $\vec{P_3}$. Let $V(\vec{P_3}) = \{v_1, v_2, v_3\}$ and the arcs of $\vec{P_3}$ be $e_i = (v_i, v_{i+1})$ for $1 \leq i \leq 2$. Then (e_1, e_2); (v_i, v_{i+1}) for $1 \leq i \leq 2$; (e_i, v_{i+1}) for $1 \leq i \leq 2$; and (v_i, e_i) for $1 \leq i \leq 2$ are the arcs of $T(A_r)$. The directed path number of $\vec{P_3}$ is one, say P. Then the directed pathos vertex P is a neighbor of the vertices e_1 and e_2. This shows that the crossing number of $DPT(A_r)$ is zero, i.e., $\mathrm{cr}(DPT(A_r)) = 0$ (see Figure 4). Hence $DPT(A_r)$ is planar.

Case 3. Suppose that the underlying graph of A_r is $K_{1,3}$. Let $V(A_r) = \{v_1, v_2, v_3, v_4\}$ and $A(A_r) = \{e_1, e_2, e_3\}$ such that v_1 and $e_1 = (v_1, v_2)$ are the root and root arc of A_r, respectively, and $e_i = (v_2, v_{i+1})$ for $2 \leq i \leq 3$. Then (e_1, e_2); (e_1, e_3); (v_1, v_2); (v_2, v_{i+1}) for $2 \leq i \leq 3$; (e_i, v_{i+1}) for $1 \leq i \leq 3$; (v_2, e_2); and (v_2, e_3) are the arcs of $T(A_r)$. Let $P(A_r) = \{P_1, P_2\}$ be a directed pathos set of A_r such that P_1 lies on the arcs $(v_1, v_2), (v_2, v_3)$ and P_2 lies on (v_2, v_4). Then the directed pathos vertex P_1 is a neighbor of the vertices v_1v_2, v_2v_3, P_2 and P_2 is a neighbor of v_2v_4. This shows that the crossing number of $DPT(A_r)$ is zero (see Figure 3). Thus $DPT(A_r)$ is planar. This completes the proof. $\square$

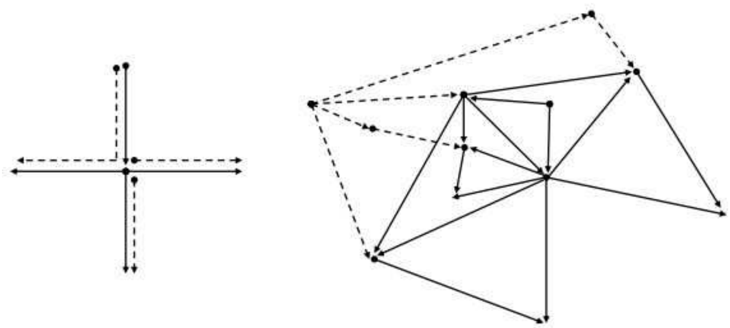

Figure 3.

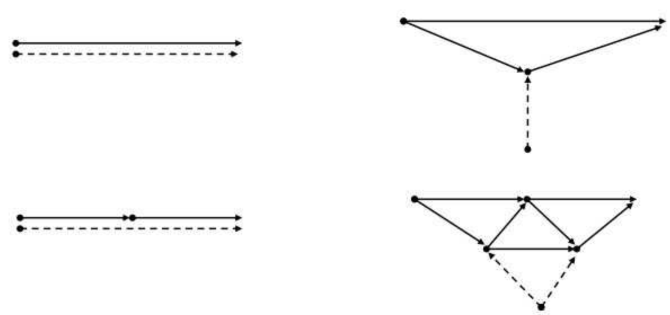

Figure 4.

We now establish a characterization of digraphs whose $DPT(A_r)$ are outerplanar; maximal outerplanar; and minimally nonouterplanar.

Theorem 6.2. *A directed pathos total digraph $DPT(A_r)$ of an arborescence A_r is outerplanar if and only if A_r is either $\vec{P}_2$ or $\vec{P}_3$.*

Proof. Suppose that $DPT(A_r)$ is outerplanar. Assume that $A_r = \vec{P}_4$. Let $V(\vec{P}_4) = \{v_1, v_2, v_3, v_4\}$ and the arcs of $\vec{P}_4$ be $e_i = (v_i, v_{i+1})$ for $1 \leq i \leq 3$. Then (e_1, e_2); (e_2, e_3); (v_i, v_{i+1}) for $1 \leq i \leq 3$; (e_i, v_{i+1}) for $1 \leq i \leq 3$; and (v_i, e_i) for $1 \leq i \leq 3$ are the arcs of $T(A_r)$. The directed path number of $\vec{P}_4$ is one, say P. Then the directed pathos vertex P is a neighbor of the vertices

e_1, e_2, and e_3. This shows that the inner vertex number of $DPT(A_r)$ is one, i.e., $i(DPT(A_r)) = 1$ (see Figure 5), a contradiction.
Conversely, suppose that A_r is either $\vec{P_2}$ or $\vec{P_3}$. If A_r is $\vec{P_2}$, then the underlying graph of $DPT(A_r)$ is $K_{1,3} + e$. Clearly $i(DPT(A_r)) = 0$. Thus $DPT(A_r)$ is outerplanar. On the other hand, if A_r is $\vec{P_3}$, then Case 2 of sufficiency of Theorem 6.1 implies that the crossing number of $DPT(A_r)$ is zero. This also shows that the inner vertex number of $DPT(A_r)$ is zero, i.e., $i(DPT(A_r)) = 0$ (see Figure 4). Hence $DPT(A_r)$ is outerplanar. This completes the proof. □

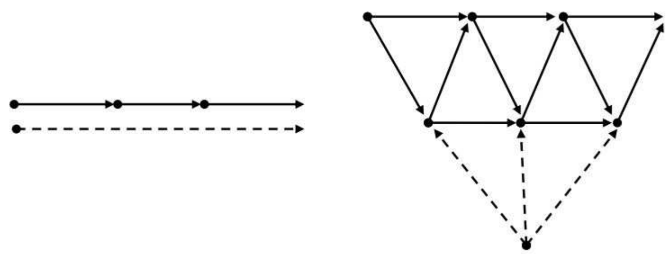

Figure 5.

Theorem 6.3. *A directed pathos total digraph $DPT(A_r)$ of an arborescence A_r is maximal outerplanar if and only if A_r is $\vec{P_3}$.*

Proof. Suppose that $DPT(A_r)$ is maximal outerplanar. We consider the following cases.
Case 1. Assume that the total degree of each vertex of A_r is at least four, i.e., $td(v) \geq 4$, for every vertex $v \in A_r$. By Theorem 6.1, $DPT(A_r)$ is nonplanar, a contradiction.
Case 2. If there exists a vertex of total degree three in A_r. By Theorem 6.2, $DPT(A_r)$ is nonouterplanar, a contradiction.
Case 3. If $A_r = \vec{P_2}$, then the underlying graph of $DPT(A_r)$ is $K_{1,3}+e$. Clearly $i(DPT(A_r)) = 0$. Thus $DPT(A_r)$ is outerplanar. Furthermore, since the addition of an arc does not alter the outerplanarity of $DPT(A_r)$, it follows that $DPT(A_r)$ is not maximal outerplanar, a contradiction.
Case 4. If $A_r = \vec{P}_{n+3}$ $(n \geq 1)$, then the inner vertex number of the corresponding $DPT(A_r)$ equals n. Clearly, $DPT(A_r)$ is nonouterplanar, again a contradiction.
Conversely, suppose that $A_r = \vec{P_3}$. By Proposition 5.12, the order and size of $DPT(A_r)$ are $n = 6$ and $m = 9$, respectively. But $m = 9 = 2n - 3$. Since the size of $DPT(A_r)$ is nine, Theorem 2.1 implies that $DPT(A_r)$ is maximal outerplanar. This completes the proof. □

Theorem 6.4. *A directed pathos total digraph $DPT(A_r)$ of an arborescence A_r is minimally nonouterplanar if and only if A_r is $\vec{P_4}$.*

Proof. Suppose that $DPT(A_r)$ is minimally nonouterplanar. Assume that $A_r = \vec{P_5}$. By Case 4 of necessity of Theorem 6.3, $i(DPT(A_r)) = 2$ (see Figure.2), a contradiction.
Conversely, suppose that $A_r = \vec{P_4}$. By Case 4 of necessity of Theorem 6.3, $i(DPT(A_r)) = 1$ (see Figure 5). Hence $DPT(A_r)$ is minimally nonouterplanar. This completes the proof. □

Theorem 6.5. *A directed pathos total digraph $DPT(A_r)$ of an arborescence A_r has crossing number one if and only if the underlying graph of A_r is $K_{1,4}$.*

Proof. Suppose $DPT(A_r)$ has crossing number one. Assume that the underlying graph of A_r is $K_{1,n}$ $(n \geq 5)$. Suppose $A_r = K_{1,5}$. Let $V(A_r) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $A(A_r) = \{e_1, e_2, e_3, e_4, e_5\}$ such that v_1 and $e_1 = (v_1, v_2)$ are the root and root arc of A_r, respectively; and $e_i = (v_2, v_{i+1})$ for $2 \leq i \leq 5$. Then (e_1, e_i) for $2 \leq i \leq 5$; (v_1, v_2); (v_2, v_i) for $3 \leq i \leq 6$; (e_i, v_{i+1}) for $1 \leq i \leq 5$; (v_1, e_1); and (v_2, e_i) for $2 \leq i \leq 5$ are the arcs of $T(A_r)$. Let $P(A_r) = \{P_1, P_2, P_3, P_4\}$ be a directed pathos set of A_r such that P_1 lies on the arcs $(v_1, v_2), (v_2, v_3)$; P_2 lies on (v_2, v_4); P_3 lies on (v_2, v_5); and P_4 lies on (v_2, v_6). Then the directed pathos vertex P_1 is a neighbor of the vertices $v_1v_2, v_2v_3, P_2, P_3, P_4$; P_2 is a neighbor of v_2v_4; P_3 is a neighbor of v_2v_5; and P_4 is a neighbor of v_2v_6. This shows that the crossing number of $DPT(A_r)$ is more than one, i.e., $\mathrm{cr}(DPT(A_r)) > 1$ (see Figure 6), a contradiction.
Conversely, suppose that the underlying graph of A_r is $K_{1,4}$. By necessity of Theorem 6.1, the crossing number of $DPT(A_r)$ is one. This completes the proof. □

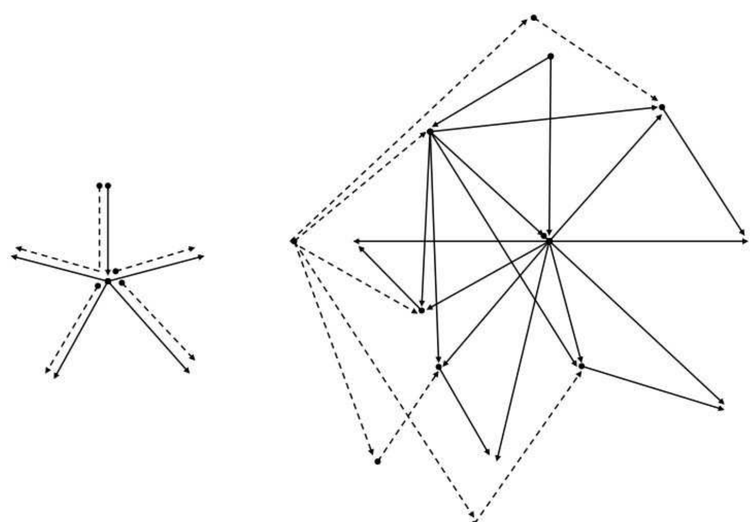

Figure 6.

Competing Interests

The authors declares that there is no competing interests regarding the publication of this paper.

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