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COMPLETE MONOTONICITY PROPERTIES OF A FUNCTION INVOLVING THE POLYGAMMA FUNCTION

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Abstract. In this paper, we study completete monotonicity properties of certain functions associated with the polygamma functions. Subsequently, we deduce some inequalities involving difference of polygamma functions.

Index Terms: Polygamma function; complete monotonicity; inequality.

1. Introduction

The classical Gamma function, which is an extension of the factorial notation to noninteger values is usually defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, \quad x > 0,$$

and satisfying the basic property

$$\Gamma(x+1) = x\Gamma(x), \quad x > 0.$$

Its logarithmic derivative, which is called the Psi or digamma function is defined as (see [1] and [2])

$$\begin{aligned} \psi(x) = \frac{d}{dx} \ln \Gamma(x) &= -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}\, dt, \quad x > 0, \qquad (1) \\ &= -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty} \frac{x}{k(k+x)}, \quad x > 0, \end{aligned}$$

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where $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) = 0.577215664...$ is the Euler-Mascheroni's constant. Derivatives of the Psi function, which are called polygamma functions are given as [1]

$$\psi^{(n)}(x) = (-1)^{n+1}\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}\,dt, \quad x>0, \tag{2}$$
$$= (-1)^{n+1} n! \sum_{k=0}^{\infty}\frac{1}{(k+x)^{n+1}}, \quad x>0,$$

satisfying the functional equation [1]

$$\psi^{(n)}(x+1) = \psi^{(n)}(x) + \frac{(-1)^n n!}{x^{n+1}}, \quad x>0, \tag{3}$$

where $n \in \mathbb{N}_0$ and $\psi^{(0)}(x) \equiv \psi(x)$. Here, and for the rest of this paper, we use the notations: $\mathbb{N} = \{1,2,3,4,\dots\}$, $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and $\mathbb{R} = (-\infty,\infty)$. Also, it is well known in the literature that the integral

$$\frac{n!}{x^{n+1}} = \int_0^\infty t^n e^{-xt}\,dt, \tag{4}$$

holds for $x>0$ and $n\in\mathbb{N}_0$. See for instance [1]. In [3], Qiu and Vuorinen established among other things that the function

$$h_1 = \psi\left(x+\frac{1}{2}\right) - \psi(x) - \frac{1}{2x}, \tag{5}$$

is strictly decreasing and convex on $(0,\infty)$. Motivated by this result, Mortici [4] proved a more generalized and deeper result which states that, the function

$$f_a = \psi(x+a) - \psi(x) - \frac{a}{x}, \quad a\in(0,1), \tag{6}$$

is strictly completely monotonic on $(0,\infty)$. Recall that a function $f:(0,\infty)\to\mathbb{R}$ is said to be completely monotonic on $(0,\infty)$ if f has derivatives of all order and $(-1)^n f^{(n)}(x) \geq 0$ for all $x\in(0,\infty)$ and $n\in\mathbb{N}_0$.
In this paper, the objective is to extend Mortici's results to the polygamma functions.

2. Some Lemmas

In order to establish our main results, we need the following lemmas.

Lemma 2.1. *Let a function $q_{\alpha,\beta}(t)$ be defined as*

$$q_{\alpha,\beta}(t) = \begin{cases} \frac{e^{-\alpha t}-e^{-\beta t}}{1-e^{-t}}, & t\neq 0,\\ \beta-\alpha, & t=0, \end{cases} \tag{7}$$

where α, β are real numbers such that $\alpha\neq\beta$ and $(\alpha,\beta)\notin\{(0,1),(1,0)\}$. Then $q_{\alpha,\beta}(t)$ is increasing on $(0,\infty)$ if and only if $(\beta-\alpha)(1-\alpha-\beta)\geq 0$ and $(\beta-\alpha)(|\alpha-\beta|-\alpha-\beta)\geq 0$.

Proof. See [5], [6] or [7]. □

Lemma 2.2. *Let $a \in (0,1)$. Then the inequality*

$$a < \frac{1-e^{-at}}{1-e^{-t}} < 1, \tag{8}$$

holds for $t \in (0,\infty)$.

Proof. Note that the function $h(t) = \frac{1-e^{-at}}{1-e^{-t}}$ which is obtained from Lemma 2.1 by letting $\alpha = 0$ and $\beta = a \in (0,1)$ is increasing on $(0,\infty)$. Also,

$$\lim_{t\to 0^+} h(t) = a \quad \text{and} \quad \lim_{t\to\infty} h(t) = 1.$$

Then for $t \in (0,\infty)$, we have

$$a = \lim_{t\to 0^+} h(t) = h(0) < h(t) < h(\infty) = \lim_{t\to\infty} h(t) = 1,$$

which gives inequality (8). □

3. Main Results

We now present our results in this section.

Theorem 3.1. *Let $f_{a,k}(x)$ and $h_{a,r}(x)$ be defined for $a \in (0,1)$, $k \in \{2s : s \in \mathbb{N}_0\}$, $r \in \{2s+1 : s \in \mathbb{N}_0\}$ and $x \in (0,\infty)$ as*

$$f_{a,k}(x) = \psi^{(k)}(x+a) - \psi^{(k)}(x) - \frac{ak!}{x^{k+1}}, \tag{9}$$

and

$$h_{a,r}(x) = \psi^{(r)}(x+a) - \psi^{(r)}(x) - \frac{ar!}{x^{r+1}}. \tag{10}$$

Then $f_{a,k}(x)$ and $-h_{a,r}(x)$ are strictly completely monotonic on $(0,\infty)$.

Proof. By repeated differentiations with respect to x, and by using (2) and (4), we obtain

$$\begin{aligned} f_{a,k}^{(n)}(x) &= \psi^{(k+n)}(x+a) - \psi^{(k+n)}(x) - \frac{(-1)^n a(k+n)!}{x^{k+n+1}} \\ &= (-1)^{k+n+1}\int_0^\infty \frac{t^{k+n}e^{-(x+a)t}}{1-e^{-t}}\,dt - (-1)^{k+n+1}\int_0^\infty \frac{t^{k+n}e^{-xt}}{1-e^{-t}}\,dt \\ &\quad - (-1)^n a\int_0^\infty t^{k+n}e^{-xt}\,dt. \end{aligned}$$

This implies that

$$\begin{aligned} (-1)^n f_{a,k}^{(n)}(x) &= -\int_0^\infty \frac{t^{k+n}e^{-xt}e^{-at}}{1-e^{-t}}\,dt + \int_0^\infty \frac{t^{k+n}e^{-xt}}{1-e^{-t}}\,dt - a\int_0^\infty t^{k+n}e^{-xt}\,dt \\ &= \int_0^\infty \left[\frac{1-e^{-at}}{1-e^{-t}} - a\right] t^{k+n}e^{-xt}\,dt \\ &> 0, \end{aligned}$$

which is as a result of Lemma 2.2. Alternatively, we could proceed as follows.

$$\begin{aligned}
(-1)^n f_{a,k}^{(n)}(x) &= \int_0^\infty \left[\frac{1-e^{-at}}{1-e^{-t}} - a\right] t^{k+n} e^{-xt}\, dt \\
&= a\int_0^\infty \left[\frac{1-e^{-at}}{at} - \frac{1-e^{-t}}{t}\right] \frac{t^{k+n+1}e^{-xt}}{1-e^{-t}}\, dt \\
&> 0.
\end{aligned}$$

Notice that, since the function $\frac{1-e^{-t}}{t}$ is strictly decreasing on $(0,\infty)$, then for $a \in (0,1)$, we have $\frac{1-e^{-at}}{at} > \frac{1-e^{-t}}{t}$. Hence $f_{a,k}(x)$ is strictly completely monotonic on $(0,\infty)$. Similarly, we have

$$\begin{aligned}
-h_{a,r}^{(n)}(x) &= \frac{(-1)^n a(r+n)!}{x^{r+n+1}} + \psi^{(r+n)}(x) - \psi^{(r+n)}(x+a) \\
&= (-1)^n a \int_0^\infty t^{r+n} e^{-xt}\, dt + (-1)^{r+n+1}\int_0^\infty \frac{t^{r+n}e^{-xt}}{1-e^{-t}}\, dt \\
&\quad - (-1)^{r+n+1}\int_0^\infty \frac{t^{r+n}e^{-(x+a)t}}{1-e^{-t}}\, dt,
\end{aligned}$$

which implies that

$$\begin{aligned}
(-1)^n \left(-h_{a,r}\right)^{(n)}(x) &= a\int_0^\infty t^{r+n}e^{-xt}\, dt + \int_0^\infty \frac{t^{r+n}e^{-xt}}{1-e^{-t}}\, dt - \int_0^\infty \frac{t^{r+n}e^{-xt}e^{-at}}{1-e^{-t}}\, dt \\
&= \int_0^\infty \left[a + \frac{1-e^{-at}}{1-e^{-t}}\right] t^{r+n}e^{-xt}\, dt \\
&> 0.
\end{aligned}$$

Hence $-h_{a,r}(x)$ is strictly completely monotonic on $(0,\infty)$. □

Remark 3.2. Since every completely monotonic function is convex and decreasing, it follows that $f_{a,k}(x)$ is strictly convex and strictly decreasing on $(0,\infty)$. In this way, $h_{a,r}(x)$ is strictly concave and strictly increasing on $(0,\infty)$.

Corollary 3.3. *The inequality*

$$\frac{ak!}{x^{k+1}} < \psi^{(k)}(x+a) - \psi^{(k)}(x) < \psi^{(k)}(a) - \psi^{(k)}(1) + k!\left(\frac{a}{x^{k+1}} + \frac{1}{a^{k+1}} - a\right), \quad (11)$$

holds for $a \in (0,1)$, $k \in \{2s : s \in \mathbb{N}_0\}$ *and* $x \in (1,\infty)$.

Proof. Since $f_{a,k}(x)$ is decreasing, then for $x \in (1,\infty)$ and by applying (3), we obtain

$$\begin{aligned}
0 = \lim_{x\to\infty} f_{a,k}(x) < f_{a,k}(x) < f_{a,k}(1) &= \psi^{(k)}(a+1) - \psi^{(k)}(1) - ak! \\
&= \psi^{(k)}(a) - \psi^{(k)}(1) + \frac{k!}{a^{k+1}} - ak!,
\end{aligned}$$

which completes the proof. □

Remark 3.4. In particular, if $a = \frac{1}{2}$ and $k = 0$ in Corollary 3.3, then we obtain

$$\frac{1}{2x} < \psi\left(x + \frac{1}{2}\right) - \psi(x) < \frac{1}{2x} + \frac{3}{2} - 2\ln 2, \quad x \in (1, \infty). \tag{12}$$

Also, if $a = \frac{1}{2}$ and $k = 2$ in Corollary 3.3, then we obtain

$$\frac{1}{x^3} < \psi''\left(x + \frac{1}{2}\right) - \psi''(x) < \frac{1}{x^3} + 15 - 12\zeta(3), \quad x \in (1, \infty), \tag{13}$$

where $\zeta(x)$ is the Riemann zeta function.

Corollary 3.5. *The inequality*

$$\psi^{(r)}(a) - \psi^{(r)}(1) + r!\left(\frac{a}{x^{r+1}} - \frac{1}{a^{r+1}} - a\right) < \psi^{(r)}(x+a) - \psi^{(r)}(x) < \frac{ar!}{x^{r+1}}, \tag{14}$$

holds for $a \in (0, 1)$, $r \in \{2s + 1 : s \in \mathbb{N}_0\}$ *and* $x \in (1, \infty)$.

Proof. Likewise, since $h_{a,r}(x)$ is increasing, then for $x \in (1, \infty)$, we obtain

$$\psi^{(r)}(a) - \psi^{(r)}(1) - \frac{r!}{a^{r+1}} - ar! = h_{a,r}(1) < h_{a,r}(x) < \lim_{x\to\infty} h_{a,r}(x) = 0,$$

which yields (14). □

Remark 3.6. If $a = \frac{1}{2}$ and $r = 1$ in Corollary 3.5, then we obtain

$$\frac{1}{2x^2} + \frac{\pi^2}{3} - \frac{9}{2} < \psi'\left(x + \frac{1}{2}\right) - \psi'(x) < \frac{1}{2x^2}, \quad x \in (1, \infty). \tag{15}$$

Furthermore, if $a = \frac{1}{2}$ and $r = 3$ in Corollary 3.5, then we obtain

$$\frac{3}{x^4} + \frac{14\pi^4}{15} - 99 < \psi'''\left(x + \frac{1}{2}\right) - \psi'''(x) < \frac{3}{x^4}, \quad x \in (1, \infty). \tag{16}$$

Remark 3.7. If $k = 0$ in Theorem 3.1, then we obtain the main results of [4] as a special case of the present results.

Remark 3.8. This paper is a modified version of the preprint [8].

Competing Interests

The author declares that there are no competing interests regarding the publication of this paper.

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