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Scaled Prediction Variances of Equiradial Design under Changing Design Sizes, Axial Distances and Center Runs

Bartholomew, Desmond Chekwube[1*], Obite, Chukwudi Paul[1] and Ismaila-Cosmos Joan[2]

[1]Department of Statistics, Federal University of Technology, Owerri, Nigeria.
[2]Department of Mathematics and Statistics, University of Port Harcourt, Nigeria.

Authors' contributions

This work was carried out in collaboration among all authors. Author BDC designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. Authors OCP and ICJ managed the literature searches and references. All authors read and approved the final manuscript.

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Abstract

The aim of every design choice is to minimize the prediction error, especially at every location of the design space, thus, it is important to measure the error at all locations in the design space ranging from the design center (origin) to the perimeter (distance from the origin). The measure of the errors varies from one design type to another and considerably the distance from the design center. Since this measure is affected by design sizes, it is ideal to scale the variance for the purpose of model comparison. Therefore, we have employed the Scaled Prediction Variance and D – optimality criterion to check the behavior of equiradial designs and compare them under varying axial distances, design sizes and center points. The following similarities were observed: (i) increasing the design radius (axial distance) of an equiradial design changes the maximum determinant of the information matrix by five percent of the new axial distance (5% of 1.414 = 0.07) see Table 3. (ii) increasing the n_c center runs pushes the maximum $SPV_{(\tilde{x})}$ to the furthest distance from the design center (0 0) (iii) changing the design radius changes the location in the design region with maximum $SPV_{(\tilde{x})}$ by a multiple of the change and (iv) changing the design radius also does not change the maximum $SPV_{(\tilde{x})}$ at different radial points and center runs .

*Corresponding author: E-mail: desmond.bartholomew@futo.edu.ng;

Based on the findings of this research, we therefore recommend consideration of equiradial designs with only two center runs in order to maximize the determinant of the information matrix and minimize the scaled prediction variances.

Keywords: Axial distance; center point; equiradial design; error; scaled prediction variance.

1 Introduction

In design of experiment, several factors affect the performance of a chosen design type, especially when the goal of any design is to estimate the model parameters without bias and with minimum variance. These factors may include model type, design region, positioning of the design points and missing design points. Given that we are interested in minimizing the variance of the regression coefficients, that is $Var\left(\hat{\beta}_i\right),\quad i=0,\ 1,\ 2,\ldots,k$, for k regressors, then the key factor in choosing the design is the concept of orthogonality. Consider a first-order design model with design matrix as X , and a fixed sample size N , the elements on the diagonals of $\left(X'X\right)^{-1}$ are minimized by making the off-diagonals of $X'X$ zero and forcing the diagonals of $X'X$ to be as large as possible. This is achieved by using variance-optimal first order designs such as the 2^k factorial designs and 2^{k-p} factorial fractions of resolution III and above. The design matrix X is a function of the location in the design variables at which one predicts and also a function of the model as well as the design.

The prediction variance $Var\left(\hat{y}\left(\breve{x}\right)\right)$, read as "the variance of the predicted y value at design location $\breve{x}$ " varies from location to location in the design space and gives a reflection of how well one predicts with the given model. The scaled prediction variance is especially important where models are used for process optimization, such as in response surface methodology, Myers et al. [1]. It used when comparing designs and as such it is often convenient to scale the prediction variance, that is, to work with the scaled prediction variance. By scaling, the design size N is used to multiply the prediction variance and the result divided by the mean square error σ^2 of the supposed design. The multiplication by N allows the quantity to reflect variance on a per observation basis and division by σ^2 makes the quantity scale-free. The scaled prediction variance gives a measure of the error at all locations in the design space ranging from the design center to the perimeter and varies from one design type to another.

Equiradial design are special type of two-factor designs that consists of two or more sets of design points where the points for each set have the same distance from the design origin, Khuri and Cornel [2]. They are usually defined on a common sphere for modeling second-order response function. An equiradial design for a two variable has a set of five points defined on a circle of radius (axial distance) $\rho \geq 1$ from the design center. The center point (or points) forms a second set in a circle of radius zero, Khuri and Cornel [2]. The rotable central composite design is a member of the larger class of equiradial designs, Box and Wilson [3]. For example, with $k=2$ and $\alpha=\sqrt{2}$, the 2^2 full factorial points and the four axial (star) points form a set of eight points on the circle of radius $\rho=\sqrt{2}$ and the center point (or points) forms a second set on a circle of radius zero. Hence, the inscribed central composite design and the rotable central composite design automatically belong to the larger class of equiradial designs.

In this paper, the aim is to determine the Scaled Prediction Variances of equiradial design defined on design regions under changing design sizes, axial distances and center runs. The following research objectives are considered:

(i) To examine the behaviour of the Scaled Prediction Variances of spherical equiradial designs for changing design sizes.

(ii) To examine the behaviour of the Scaled Prediction Variances of spherical equiradial designs for changing center points

(iii) To compare the behavior of the designs with axial distances $\rho = 1$ *and* $\rho = \sqrt{2}$ of spherical equiradial designs on similar design regions.

A simple layout comprising the radial points (n_1), the center points (n_c), the design size (N), the determinant of information matrix (det (M)), the Scaled Prediction Variances for all the design points in the design region, the maximum scaled prediction variance (SPVmax) with the corresponding location in the design region and rank of the determinant of the M (information matrix). Each layout component has been well explained in Myers et al. [1], which serves as a major reference in Response Surface Methodology.

This paper is divided into six sections. Section one above gives the general introduction of the work. Section two is devoted to literature review. Section three takes care of materials and methods. Section four is results and discussion. Section five treats six treats conclusion and recommendations of the work. Following Section five are references.

2 Literature Review

Atkinson and Donev [4] investigated several design regions in optimal design theory and include spherical, cuboidal, simplex and also irregular regions. The composition of the several design points and their corresponding model type is generally a factor of some axial distances which specifies the nature of the design geometry. Three commonly encountered axial distances have been explored in Iwundu [5]. It is important to make a design as robust as possible in order to exploit the interaction between control and uncontrollable noise variables and find the settings of the control factors that minimize response variation from uncontrolled factors. In experimental design, there are known sources of variability that may be controlled by the experimenter (control variables), there are still some more influential factors that may not be controlled by the experimenter (uncontrollable noise variables). These uncontrollable noise variables are classed as error components. The three basic principles of experimentation namely randomization, replication and local control (blocking) are complementary to each other in increasing the design efficiency by minimizing the effect of these error components, Preece [6]. Therefore the principles of experimentation are highly recommended during the construction and implementation of equiradial designs. Bartholomew [7], studied the effects of different types of fertilizers and soil types on the yield of tiger nuts and the marginal effects of the error components were revealed in the two way multiple analysis of variance. Iwundu [8] considered the behavior of alternative second-order N-point equiradial designs under variations of model parameters for design radius $\rho = 1$ and established relationships among some alphabetic optimality criteria with regards to the designs and the models. It is important to note that equiradial designs for $\rho = 1$ *and* $\rho = \sqrt{2}$, $n_1 = 5, 6, 7, 8$ *when* $n_c \geq 1$ has been studied by Iwundu and Onu [9] and the results compared with central component design using some alphabetical based criteria. However, we need to have results for higher radial points (from 8 and above) and ascertain the behavior of the equiradial design at these radial point for changing design sizes and center runs. On this gap, we therefore commenced this research work.

3 Materials and Methods

Let equation (1) be the predicted response for the dependent variable y

$$\hat{y} = X\hat{\beta} \tag{1}$$

Let the point $\breve{x}_0$ be defined as

$\breve{x}_0 = \begin{pmatrix} 1 \\ x_{01} \\ x_{02} \\ . \\ . \\ . \\ x_{0k} \end{pmatrix}$ be a point in X for a first order main effect model in k variables.

The mean response at $\breve{x}_0$ is

$$\mu_{y|x_0} = \hat{\beta}_0 + \hat{\beta}_1 x_{01} + \hat{\beta}_2 x_{02} + \ldots + \hat{\beta}_k x_{0k} = \breve{x}_0 \hat{\beta} \tag{2}$$

where $\hat{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ . \\ . \\ . \\ . \\ \beta_k \end{pmatrix}$

The estimated mean response at $\breve{x}_0$ is

$$\hat{y}(\breve{x}_0) = x_0' \hat{\beta} \tag{3}$$

This estimator in equation (3) is unbiased and the corresponding variance is called the predictive variance or the variance of at $\breve{x}_0$ defined as

$$Var\left[\hat{y}(\breve{x}_0)\right] = \sigma^2 \breve{x}_0' \left(X'X\right)^{-1} \breve{x}_0 \tag{4}$$

But when comparing designs, it is often convenient to scale the variance, that is, to work with the scaled predictive variance as given by

$$SPV_{(\breve{x})} = \frac{N\,Var\left[\hat{y}(\breve{x}_0)\right]}{\sigma^2} = \frac{N\sigma^2 \breve{x}_0' \left(X'X\right)^{-1} \breve{x}}{\sigma^2} = \breve{x}_0' \left(M\right)^{-1} \breve{x} \tag{5}$$

The division by σ^2 make the quantity scale free and multiplication by N allows the quantity to reflect variances on a per observation basis and removes the effect of varying design sample sizes.

Consider the design D_9 for $\left(\rho = 1 \quad and \quad \rho = \sqrt{2}\right)$ and its design matrix, X , constructed using same procedures as for the 6-point equiradial design expressed in Khuri and Cornel [2], Myers et al. [1] and Iwundu [5]. The 9-point equiradial design with $\rho = 1$ and $n_c = 1$ center point is

$$D_9 = \begin{pmatrix} 1 & 0 \\ 0.71 & 0.71 \\ 0 & 1 \\ -0.71 & 0.71 \\ -1 & 0 \\ -0.71 & -0.71 \\ 0 & -1 \\ 0.71 & -0.71 \\ 0 & 0 \end{pmatrix} \quad when \ \rho = 1 \qquad D_9 = \begin{pmatrix} 1.414 & 0 \\ 1.000 & 1.000 \\ 0.000 & 1.414 \\ -1.000 & 1.000 \\ -1.414 & 0.000 \\ -1.000 & -1.000 \\ 0 & -1.414 \\ 1.414 & -1.414 \\ 0 & 0 \end{pmatrix} \quad when \ \rho = \sqrt{2}$$

The design measures associated with the equiradial designs are such that the design matrices are expressed as

$$\left\{ \begin{matrix} x_a & & x_b \\ \rho\, Cos\left(\theta + \dfrac{2\pi u}{n_1}\right) & & \rho\, Sin\left(\theta + \dfrac{2\pi u}{n_1}\right) \end{matrix} \right\} \qquad u = 0, 1, 2, \dots\, n_1 - 1$$

where

x_a and x_b are two controllable variables,
ρ is the radius(axial distance) of the design,
n_1 is the number of points on the sphere
n_c represents c center points in the design region.

This study focuses on the following designs $\rho = 1 \ \ and \ \ \rho = \sqrt{2}$, $n_1 = 8, 9, 10, 11 \ \ when \ \ n_c \geq 1$, the value of θ is assumed equal to zero since θ has no effect on the information matrix, $X'X$, of the design, Myers et al. [1]. Again, recall $Radian\ (r) = \dfrac{180}{\pi}$, therefore $\pi = 180, \ when \quad r = \rho = 1, \ and \ \pi = \dfrac{180}{\sqrt{2}} \ when \quad r = \rho = \sqrt{2}$.

For the design matrix, X, the associated information matrix, $X'X$, is assumed nonsingular and normalized as $M = \dfrac{X'X}{N}$, where N is the design size and M is symmetric. The inverse of the moment matrix M^{-1} is unique and gives the matrix of estimates of variances and covariances of model coefficients. The inverse matrix is cardinal for design comparison purposes as most optimality criteria, as in Rady et al. [10], are defined as functional of M^{-1}. The determinant of information matrix as well as the Scaled Prediction Variances associated with each design location for the two design radii (axial distances) shall be computed. The determinant of the information matrix will be compared using the D – optimality criterion. The D – optimality criterion is one of the popular alphabetic optimality criteria that is used to determine an ideal design that would minimize the generalized variance of the parameter estimates and also minimize variance of predicted model over the design region. The D stands for determinant which means D-optimality is one in which the determinant of the moment matrix $M = \frac{X'X}{N}$ is maximized over all the designs and equivalently minimizes the determinant of the variance-covariance matrix. According to John and Draper [11], D-optimality criterion is given by

$$\phi(M(\xi)) = Max\{\det M(\xi)\} = Min\{\det M^{-1}(\xi)\} \tag{6}$$

Where

det (.) is determinant of matrix.
$M(\xi)$ is moment matrix of design ξ
M^{-1} is the inverse of the moment matrix

The 10-point equiradial design for $\rho = 1.0$, $n_1 = 9$ and $n_c = 1$ contains the design points (x_a x_b) as (1, 0), (0.77, 0.64), (0.17, 0.98), (-0.50, 0.87), (-0.94, 0.34), (-0.94, -0.34), (-0.50, -0.87), (0.17, -0.98), (0.77, -0.64), (0, 0). The 11-point equiradial design for $\rho = 1.0$, $n_1 = 10$ and $n_c = 1$ contains the design points (1, 0), (0.81, 0.59), (0.31, 0.95), (-0.31, 0.95), (-0.81, 0.59), (-1, 0), (-0.81, -0.59), (-0.31, -0.95), (0.31, -0.95), (0.81, -0.59), (0, 0). The 12-point equiradial design for $\rho = 1.0$, $n_1 = 11$ and $n_c = 1$ contains the design points (1, 0), (0.84, 0.54), (0.41, 0.91), (-0.14, 0.99), (-0.65, 0.76), (-0.96, 0.28), (-0.96, -0.28), (-0.65, -0.76), (-0.14, -0.99), (0.41, -0.91), (0.84, -0.54), (0, 0), Myers et al. [1]. The corresponding equiradial design for $\rho = \sqrt{2}$ are obtained by multiplying the various x_a *and* x_b locations in the design points in $\rho = 1$ by $\sqrt{2}$ or following the standard method of constructing equiradial designs.

4 Results and Discussion

The computations were done manually with the use of Microsoft Excel Matrix computations for equation 5. We are studying equiradial design with radial points $n_1 = 8, 9, 10, 11$ because the scaled prediction variances for equiradial design with radial points $n_1 = 5, 6, 7, 8$ has been done by Iwundu and Onu [9]. We included the 8 radial point in our research in order to compare our results with their findings before extending to 9, 10 and 11 radial points.

4.1 Spherical equiradial designs $(\rho = 1.0,\ n_c \in [1 \quad 5])$

The computations for equiradial design involving $n_1 = 8, 9, 10, 11$, $\rho = 1.0$ *and* $n_c = 1, 2, 3, 4, 5$ are as in Table 1.

Table 1. Summary of computations for spherical Equiradial designs $\left(\rho=1, \quad n_c=1,2,3,4,5\right)$

$\rho=1$	**Scaled Predictive Variance**				
Design Size N	9	10	11	12	13
Radial Point (n_1)	8				
Center Point (n_c)	1	2	3	4	5
Det (M)	0.00024	0.00026	0.00022	0.00017	0.00013
(1 0)	5.61	6.23	6.85	7.47	8.10
(0.71 0.71)	5.66	6.29	6.92	7.55	8.17
(0 1)	5.61	6.23	6.85	7.47	8.10
(-0.71 0.71)	5.65	6.28	6.91	7.53	8.16
(-1 0)	5.64	6.27	6.89	7.52	8.15
(-0.71 -0.71)	5.55	6.16	6.78	7.40	8.01
(0 -1)	5.64	6.27	6.89	7.52	8.15
(0.71 -0.71)	5.65	6.28	6.91	7.53	8.16
(0 0)	9.00	5.00	3.67	3.00	2.60
Max SPV point	(0 0)	(0.71 0.71)	(0.71 0.71)	(0.71 0.71)	(0.71 0.71)
Max SPV	9.00	6.29	6.92	7.55	8.17
Rank of Det (M)	2	1	3	4	5
	Scaled Predictive Variance				
Design Size N	10	11	12	13	14
Radial Point (n_1)	9				
Center Point (n_c)	1	2	3	4	5
Det (M)	0.000230	0.000259	0.000231	0.000190	0.000153
(1 0)	5.51	6.06	6.61	7.16	7.71
(0.77 0.64)	5.60	6.16	6.72	7.28	7.71
(0.17 0.98)	5.50	6.06	6.61	7.16	7.71
(-0.5 0.87)	5.60	6.16	6.72	7.28	7.84
(-0.94 0.34)	5.54	6.10	6.65	7.21	7.76
(-0.94 -0.34)	5.54	6.10	6.65	7.21	7.76
(-0.5 -0.87)	5.60	6.16	6.72	7.28	7.84
(0.17 -0.98)	5.50	6.06	6.61	7.16	7.71
(0.77 -0.64)	5.60	6.16	6.72	7.28	7.84
(0 0)	10.00	5.50	4.00	3.25	2.80
Max SPV	(0 0)	(-0.5 0.87)	(-0.5 0.87)	(-0.5 0.87)	(-0.5 0.87)
Max	10.0	6.2	6.7	7.3	7.8
Rank of Det (M)	3	1	2	4	5
	Scaled Prediction Variance				
Design Size N	11	12	13	14	15
Radial Point (n_1)	10				
Center Point (n_c)	1	2	3	4	5
Det (M)	0.00022	0.00026	0.00024	0.00021	0.00017
(1 0)	5.50	5.99	6.49	6.99	7.49
(0.81 0.59)	5.51	6.01	6.52	7.02	7.52
(0.31 0.95)	5.49	5.99	6.49	6.99	7.48
(-0.31 0.95)	5.49	5.99	6.49	6.99	7.48
(-0.81 0.59)	5.51	6.01	6.52	7.02	7.52

$\rho = 1$	Scaled Predictive Variance				
(-1 0)	5.50	5.99	6.49	6.99	7.49
(-0.81 -0.59)	5.51	6.01	6.52	7.02	7.52
(-0.31 -0.95)	5.49	5.99	6.49	6.99	7.48
(0.31 -0.95)	5.49	5.99	6.49	6.99	7.48
(0.81 -0.59)	5.51	6.01	6.52	7.02	7.52
(0 0)	11.00	6.00	4.33	3.50	3.00
Max SPV	(0 0)	(0.81 0.59)	(0.81 0.59)	(0.81 0.59)	(0.81 0.59)
Max	11.0	6.0	6.5	7.0	7.5
Rank of Det (M)	3	1	2	4	5
	Scale Variance				
Design Size N	12	13	14	15	16
Radial Point (n_1)	11				
Center Point (n_c)	1	2	3	4	5
Det (M)	0.0002085	0.0002580	0.0002481	0.0002187	0.0001856
(1 0)	5.47	5.92	6.38	6.83	7.29
(0.84 0.54)	5.47	5.92	6.38	6.83	7.29
(0.41 0.91)	5.43	5.88	6.34	6.79	7.24
(-0.14 0.99)	5.42	5.88	6.33	6.78	7.23
(-0.65 0.76)	5.47	5.92	6.38	6.84	7.29
(-0.96 0.28)	5.47	5.93	6.39	6.84	7.30
(-0.96 -0.28)	5.47	5.93	6.39	6.84	7.30
(-0.65 -0.76)	5.47	5.92	6.38	6.84	7.29
(-0.14 -0.99)	5.42	5.88	6.33	6.78	7.23
(0.41 -0.91)	5.43	5.88	6.34	6.79	7.24
(0.84 -0.54)	5.47	5.92	6.38	6.83	7.29
(0 0)	12.00	6.50	4.67	3.75	3.20
Max SPV	(0 0)	(0 0)	(-0.96 0.28)	(-0.96 0.28)	(-0.96 0.28)
Max	12.00	6.50	6.39	6.84	7.30
Rank of Det (M)	4	1	2	3	5

4.2 Spherical equiradial designs $\left(\rho = \sqrt{2},\ n_c \in [1\ \ 5]\right)$

The computations involving equiradial designs constructed using $n_1 = 8, 9, 10, 11$, $\rho = \sqrt{2}$ *and* $n_c = 1, 2, 3, 4, 5$ are as in Table 2.

Table 2. Summary of computations for spherical Equiradial designs $\left(\rho = \sqrt{2},\ n_c = 1,2,3,4,5\right)$

$\rho = \sqrt{2}$	Scale Variance				
Design Size N	9	10	11	12	13
Radial Point (n_1)	8				
Center Point (n_c)	1	2	3	4	5
Det (M)	0.06	0.07	0.06	0.04	0.03
(1.414 0)	5.61	6.23	6.85	7.47	8.10
(1.00 1.00)	5.66	6.29	6.92	7.55	8.17
(0.00 1.414)	5.61	6.23	6.85	7.47	8.10

$\rho=\sqrt{2}$	**Scale Variance**				
(-1.00 1.00)	5.65	6.28	6.91	7.53	8.16
(-1.414 0)	5.64	6.27	6.89	7.52	8.15
(-0.99 -0.99)	5.55	6.16	6.78	7.40	8.01
(0.00 -1.414)	5.64	6.27	6.89	7.52	8.15
(1.00 -1.00)	5.65	6.28	6.91	7.53	8.16
(0 0)	9.00	5.00	3.67	3.00	2.60
Max SPV	(0 0)	(1.00 1.00)	(1.00 1.00)	(1.00 1.00)	(1.00 1.00)
Max	9.00	6.29	6.92	7.55	8.17
Rank of Det (M)	2	1	3	4	5
	Scaled Predictive Variance				
Design Size N	10	11	12	13	14
Radial Point (n_1)	9				
Center Point (n_c)	1	2	3	4	5
Det (M)	0.06	0.07	0.06	0.05	0.04
(1.414 0)	5.51	6.06	6.61	7.16	7.71
(1.09 0.90)	5.60	6.16	6.72	7.28	7.71
(0.24 1.39)	5.50	6.06	6.61	7.16	7.71
(-0.71 1.23)	5.60	6.16	6.72	7.28	7.84
(-1.33 0.48)	5.54	6.10	6.65	7.21	7.76
(-1.33 -0.48)	5.54	6.10	6.65	7.21	7.76
(-0.71 -1.23)	5.60	6.16	6.72	7.28	7.84
(0.24 -1.39)	5.50	6.06	6.61	7.16	7.71
(1.09 -0.90)	5.60	6.16	6.72	7.28	7.84
(0 0)	10.00	5.50	4.00	3.25	2.80
Max SPV	(0 0)	(-0.71 1.23)	(-0.71 1.23)	(-0.71 1.23)	(-0.71 1.23)
Max	10.0	6.2	6.7	7.3	7.8
Rank of Det (M)	3	1	2	4	5
	Scaled Predictive Variance				
Design Size N	11	12	13	14	15
Radial Point (n_1)	10				
Center Point (n_c)	1	2	3	4	5
Det (M)	0.057	0.067	0.063	0.053	0.044
(1.414 0)	5.50	5.99	6.49	6.99	7.49
(1.15 0.83)	5.51	6.01	6.52	7.02	7.52
(0.44 1.34)	5.49	5.99	6.49	6.99	7.48
(-0.44 1.34)	5.49	5.99	6.49	6.99	7.48
(-1.15 0.83)	5.51	6.01	6.52	7.02	7.52
(-1.414 0.00)	5.50	5.99	6.49	6.99	7.49
(-1.15 -0.83)	5.51	6.01	6.52	7.02	7.52
(-0.44 -1.34)	5.49	5.99	6.49	6.99	7.48
(0.44 -1.34)	5.49	5.99	6.49	6.99	7.48
(1.15 -0.83)	5.51	6.01	6.52	7.02	7.52
(0 0)	11.00	6.00	4.33	3.50	3.00
Max SPV	(0 0)	(1.15 0.83)	(1.15 0.83)	(1.15 0.83)	(1.15 0.83)
Max	11.00	6.01	6.52	7.02	7.52
Rank of Det (M)	3	1	2	4	5
	Scaled Predictive Variance				
Design Size N	12	13	14	15	16

$\rho = \sqrt{2}$			Scale Variance		
Radial Point (n_1)	11				
Center Point (n_c)	1	2	3	4	5
Det (M)	0.05	0.07	0.06	0.06	0.05
(1.414 0)	5.47	5.92	6.38	6.83	7.29
(1.19 0.76)	5.47	5.92	6.38	6.83	7.29
(0.58 1.29)	5.43	5.88	6.34	6.79	7.24
(-0.20 1.40)	5.42	5.88	6.33	6.78	7.23
(-0.92 1.07)	5.47	5.92	6.38	6.84	7.29
(-1.36 0.40)	5.47	5.93	6.39	6.84	7.30
(-1.36 -0.40)	5.47	5.93	6.39	6.84	7.30
(-0.92 -1.07)	5.47	5.92	6.38	6.84	7.29
(-0.20 -1.40)	5.42	5.88	6.33	6.78	7.23
(0.58 -1.29)	5.43	5.88	6.34	6.79	7.24
(1.19 -0.76)	5.47	5.92	6.38	6.83	7.29
(0 0)	12.00	6.50	4.67	3.75	3.20
Max SPV	(0 0)	(0 0)	(-1.36 0.40)	(-1.36 0.40)	(-1.36 0.40)
Max	12.00	6.50	6.39	6.84	7.30
Rank of Det (M)	4	1	2	3	5

4.3 Comparison of the behavior of the Scaled Prediction Variances of equiradial designs for $\rho = 1.00$ *and* $\rho = \sqrt{2}$

The Table 3.0 takes care of objective three, where Max SPV point is the point on the design region with the highest SPV and Max is the value of the Maximum SPV.

4.4 Discussion of results

Consider Tables 1.0 and 2.0, the determinant value of information matrix increases for increasing axial distance for the corresponding design sizes and center runs. This implies that from equation (6), equiradial designs with $\rho = \sqrt{2},$ is better than equiradial designs with $\rho = 1.00$ in terms of D – optimality criterion. The Scaled Prediction Variances for corresponding design locations in the design region reduces as radial point (n_1) is increased, except for design center that was increased as radial point increase. As n_1 increases, the determinant value of information matrix decreases when the equiradial design contains only one center point. For equiradial designs having $n_c \geq 1$ center point, the determinant value of the associated information matrix was highest at two center point as shown in the rank of Det (M) column in Table 3 but decreases for increasing center runs, n_c. This holds true irrespective of the design radius ρ (axial distance). Increasing the axial distance increases the determinant of the information matrix by maximum of 0.07 units at two center points with the most minimum increase by 0.03 units at five center points. At $n_c = 1$, the maximum scaled predictive variance of associated with equiradial designs exactly equals the number of design points, N. However, $n_c = 2$ with corresponding maximum determinant of information matrix has the mini-max Scaled Prediction Variances and then increases as n_c increases. This indicates that equiradial designs give most precised when center point is two. However as radial point increases, the maximum scaled predictive variance increases at $n_c = 1$ with $(0 \quad 0)$ point and decreases at $n_c > 1$. With eight radial point, $\rho = 1$ and one center point, the maximum scaled predictive variance is at $(0 \quad 0)$point and at

Table 3. Summary of computations for spherical Equiradial designs ($\rho = 1.00$, $\rho = \sqrt{2}$ and $n_c \geq 1$

			$\rho = 1.00$				$\rho = \sqrt{2}$					
Design Size N	**Radial Point n_1**	**Center Point n_c**	**Det (M)**	**Rank**	**Max SPV point**	**Max SPV**	**Det (M)**	**Rank**	**Max SPV point**	**Max**	**Change in Det(M)**	**Max SPV Indices**
9	8	1	0.00024	2	(0 0)	9.00	0.062	2	(0.00 0.00)	9.00	0.062	100%
10		2	0.00026	1	(0.71 0.71)	6.29	0.066	1	(1.00 1.00)	6.29	0.065	100%
11		3	0.00022	3	(0.71 0.71)	6.92	0.056	3	(1.00 1.00)	6.92	0.055	100%
12		4	0.00017	4	(0.71 0.71)	7.55	0.044	4	(1.00 1.00)	7.55	0.044	100%
13		5	0.00013	5	(0.71 0.71)	8.17	0.034	5	(1.00 1.00)	8.17	0.034	100%
10	9	1	0.00023	3	(0 0)	10.00	0.059	3	(0.00 0.00)	10.00	0.058	100%
11		2	0.00026	1	(-0.5 0.87)	6.16	0.066	1	(-0.71 1.23)	6.16	0.066	100%
12		3	0.00023	2	(-0.5 0.87)	6.72	0.059	2	(-0.71 1.23)	6.72	0.059	100%
13		4	0.00019	4	(-0.5 0.87)	7.28	0.049	4	(-0.71 1.23)	7.28	0.048	100%
14		5	0.00015	5	(-0.5 0.87)	7.84	0.039	5	(-0.71 1.23)	7.84	0.039	100%
11	10	1	0.00022	3	(0 0)	11.00	0.057	3	(0 0)	11.00	0.057	100%
12		2	0.00026	1	(0.81 0.59)	6.01	0.067	1	(1.15 0.83)	6.01	0.067	100%
13		3	0.00024	2	(0.81 0.59)	6.52	0.063	2	(1.15 0.83)	6.52	0.062	100%
14		4	0.00021	4	(0.81 0.59)	7.02	0.053	4	(1.15 0.83)	7.02	0.053	100%
15		5	0.00017	5	(0.81 0.59)	7.52	0.044	5	(1.15 0.83)	7.52	0.044	100%
12	11	1	0.00021	4	(0 0)	12.00	0.053	4	(0 0)	12.00	0.053	100%
13		2	0.00026	1	(0 0)	6.50	0.066	1	(0 0)	6.50	0.066	100%
14		3	0.00025	2	(-0.96 0.28)	6.39	0.063	2	(-1.36 0.40)	6.39	0.063	100%
15		4	0.00022	3	(-0.96 0.28)	6.84	0.056	3	(-1.36 0.40)	6.84	0.056	100%
16		5	0.00019	5	(-0.96 0.28)	7.30	0.047	5	(-1.36 0.40)	7.30	0.047	100%

$(0.71 \quad 0.71)$ as center point increases. With nine radial point, $\rho = 1$ and one center point, the maximum scaled predictive variance is at $(0 \quad 0)$point and at $(-0.5 \quad 0.87)$ as center point increases. With ten radial point, $\rho = 1$ and one center point, the maximum scaled predictive variance is at $(0 \quad 0)$point and at $(0.81 \quad 0.59)$ as center point increases. With eleven radial point, $\rho = 1$ and one center point, the maximum scaled predictive variance is at $(0 \quad 0)$point and at $(-0.96 \quad 0.28)$ as center point increases. With eight radial point, $\rho = \sqrt{2}$ and one center point, the maximum scaled predictive variance is at $(0 \quad 0)$point and at $(0.71\sqrt{2} \quad 0.71\sqrt{2})$ as center point increases. With nine radial point, $\rho = \sqrt{2}$ and one center point, the maximum scaled predictive variance is at $(0 \quad 0)$point and at $(-0.5\sqrt{2} \quad 0.87\sqrt{2})$ as center point increases. With ten radial point, $\rho = \sqrt{2}$ and one center point, the maximum scaled predictive variance is at $(0 \quad 0)$point and at $(0.81\sqrt{2} \quad 0.89\sqrt{2})$ as center point increases. With eleven radial point, $\rho = 1$ and one center point, the maximum scaled predictive variance is at $(0 \quad 0)$point and at $(-0.96\sqrt{2} \quad 0.28\sqrt{2})$ as center point increases. This implies that increasing the number of center points pushes the maximum scaled predictive variance to the furthest distance from the design center and increasing the axial distance changes the design point with maximum scaled predictive variance by the number of units change in the axial distance.

5 Conclusion

The behaviours of the spherical equiradial designs under varying axial distances ρ , changing design sizes N and increased center points n_c have been examined. We therefore propose the following relationship between the equiradial designs with $\rho = 1$ and $\rho = \sqrt{2}$ as: (i) changing the axial distance changes the maximum determinant of the information matrix by five percent of the change in axial distance (ie 5% of 1.414 = 0.07) (ii) increasing the number of n_c center points (greater than one) pushes the maximum scaled predictive variance to the furthest distance from the design point (0 0) (iii) changing the axial distance changes the design point with maximum scaled predictive variance by a multiple of the change and (iv) changing the axial distance does not change the maximum value of the scaled predictive variance at different radial points and center runs

Competing Interests

Authors have declared that no competing interests exist.

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