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Checking Correctness of a Symbolic Reliability Expression for a Capacitated Network

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Authors' contributions

This work was carried out in collaboration between both authors. Author OMA envisioned and designed the study, performed the symbolic and numerical analysis, contributed to literature search and wrote the first draft of the manuscript. Author AMAR contributed to the symbolic and numerical analysis, managed literature search, wrote the proofs of the major theorems, and substantially edited the entire manuscript. Both authors read and approved the final manuscript.

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ABSTRACT

Checking a symbolic reliability expression for a flow network is useful for detecting faults in hand derivations and for debugging computer programs. This checking can be achieved in a systematic way, though it may be a formidable task. Three exhaustive tests are given when a reliability system or network has a flow constraint. These tests apply to unreliability and reliability expressions for non-coherent as well as coherent systems, and to cases when both nodes and branches are unreliable. Further properties of reliability expressions derived through various methods are discussed. All the tests and other pertinent results are proved and illustrated by examples.

Keywords: Reliability expression; exhaustive test; success state; failure state; prime implicant; Bayes decomposition; capacitated (flow) network.

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1. INTRODUCTION

When a symbolic reliability expression is obtained, whether by hand or by computer, the need arises for checking its correctness. This checking is very useful for detecting faults in hand derivations and for debugging computer programs. One way of checking a symbolic expression is to derive it by two truly independent means. However, different methods normally lead to non-identical, though equivalent, expressions. Therefore, additional work is needed to prove the equivalence of such expressions. The aim of this paper is to show that correctness of reliability expressions for flow networks can be proved in a systematic way, though indeed for long expressions it is a formidable task. Lee [1] extended the concept of terminal-pair reliability to cover the case of a flow network, namely, a network that is good if and only if a specified amount of flow can be transmitted from the input node to the output node [2-13]. Terminal-pair reliability expressions for flow or capacitated networks can be checked by any of the three exhaustive tests given in Section 4. These new tests are adaptations of some original tests given by Rushdi [14], in which reliability in a connectivity sense is checked, and which found a variety of applications [15-17]. Our new tests are not confined to the connectivity concept, as they pay due attention to the capacity constraints of the flow network. These tests might be summarized as

- Test 1, which is the analogue of the method of perfect induction in switching theory since it requires a consideration of all the states of the system. Therefore, its use is limited to small systems only.
- Test 2, which is a simplification of test 1 in which the amount of work is minimized, but the prime implicants comprising the minimal sums for either the system success or the system failure is presumed known or must be computed.
- Test 3, which handles the problem of checking an expression by breaking it down into disjoint subproblems which are more manageable and for which correctness can be verified separately.

Examples illustrate the different tests, and proofs for these tests and for other pertinent results are included. The tests apply to reliability expressions for noncoherent as well as to coherent systems. They are initially developed for the case of a system with perfectly reliable nodes, and then modified to handle node unreliability. The tests apply to a flow or capacitated network having a capacity constraint. The paper contains a list of several conditions which are necessary for the correctness of reliability expressions derived through various methods. Unless otherwise stated, the results apply to unreliability as well as to reliability expressions.

The organization of the remainder of this paper is as follows. Section 2 lists assumptions, notation and nomenclature, while Section 3 adds some preliminary definitions. Section 4 is the main contribution of this paper as it introduces the tests in ample detail, and supplements the exposition through eight demonstrative examples. Section 5 displays some further useful properties of reliability expressions, while Section 6 considers the case of imperfect nodes. Section 7 concludes the paper. An appendix presents proofs of the major theorems introduced throughout the paper.

2. ASSUMPTIONS, NOTATION, AND NOMENCLATURE

2.1 Assumptions

1) The word “system” as used here refers to the reliability block (logic) diagram, not to a schematic, or physical diagram.
2) The system is 2-state consisting of s-independent 2-state branches and nodes that can be either good or failed. The s-implies “statistically”.
3) The reliabilities of branches and nodes are not necessarily equal. Initially the nodes are considered perfectly reliable; later the effect of node unreliability is included.
4) There is no repair. All components are initially good.
5) Both source-to-terminal reliability R_{st} and overall (network) reliability R_o are considered.

2.2 Notation

n, m: numbers of branches and nodes in the logic diagram of the system.

$X_i, \overline{X}_i$: Indicator variables of successful and unsuccessful operation of branch i. These are switching random variables that take only one of the two discrete real values 0 and 1; $X_i = 1$ and $\overline{X}_i = 0$ if i is good, and $X_i = 0$ and $\overline{X}_i = 1$ if i is failed.

$N_i, \overline{N_i}$: indicator variables of successful and unsuccessful operation of node n_i.

$S, \bar{S}$: indicator variables of successful and unsuccessful operation of the system; called system success and system failure, respectively.

p_i, q_i : reliability and unreliability of branch $i : p_i \equiv \Pr\{X_i = 1\}; q_i \equiv \Pr\{\overline{X}_i = 1\} = 1 - p_i$. Both p_i and q_i take real values on the closed interval [0.0, 1.0], i.e. $0.0 \leq p_i, q_i \leq 1.0$.

p_{n_i}, q_{n_i}: reliability and unreliability of node n_i.

R, Q : reliability and unreliability of the system: $R = \Pr\{S = 1\}; Q = \Pr\{\bar{S} = 1\} = 1 - R;\ 0 \leq R,$ $Q \leq 1$. The superscripts st and o may be added to either $R\ or\ Q$ to indicate the descriptions “source-to-terminal” and “overall” respectively.

c_i: flow capacity of branch ; $c_i \geq 0$.

$C_{ij}(X)$: capacity function of (i, j) which is the maximum flow interconnection from i to j in state X that does not violate branch capacities, $C_{ij}(X) \geq 0$. For an original (i, j) : $C_{ij} = c_{ij} X_{ij}$. Since X is a switching random vector, $C_{ij}(X)$ is a discrete random variable of a probability mass function (pmf) of no more than 2^n distinct values.

C_{ij}^T: terminal- pair capacity function from node i to node j ; $C_{ij}^T \geq 0$.

X, p, c : n dimensional vectors of branch successes, reliabilities and capacities:

$$X = (X_1 X_2 \dots X_n)^T; p = (p_1 p_2 \dots p_n)^T ; \boldsymbol{c} \equiv (c_1 c_2 \dots c_n)^T.$$

T : A superscript that implies ‘transpose of a matrix or vector.’

2.3 Nomenclature

Correct expression: The expression $R(p)$ is correct if when given any valid input p , it produces the correct value for R.

A Boolean (Switching) function $\mathbf{S(X)}$: A mapping $\{0, 1\}^n \rightarrow \{0, 1\}$, i.e., $S(X)$ is any one particular assignment of the two functional values (0 or 1) for all possible 2^n values of X [18-20]. These values of X are called the states of the system described by S.

Pseudo- Boolean (Switching) function $\mathrm{C(X)}$: A mapping $\{0, 1\}^n \rightarrow R$ where R is the field of real numbers, i.e. $C(X)$ is an assignment of a real number for each of the possible 2^n values of X [4,21,12,20,22-28].

Multiaffine function of n variables $\mathbf{R(p_1, p_2 \dots, p_n)}$: An algebraic function which is a first-degree polynomial in each of its variables, i.e., if fixed values are given to any $(n - 1)$ variables, the function reduces to a first-degree polynomial in the remaining variable [14]. Multiaffine functions include: (a) Certain algebraic functions such as system reliability/ unreliability and system availability/unavailability [29]. (b) Pseudo-Boolean (switching) functions [23-28] such as source-to-terminal capacity or the squared capacity as a function of link successes.

To subsume: A logical product or term M is said to subsume another product or term P if M implies P. This occurs when all the literals of P are contained among those of M, e.g., the term $X_1 X_2 X_3$ subsumes the terms $1, X_1, X_2, X_3, X_1 X_2, X_2 X_3, X_1 X_3$, and also itself [30,31]. In a disjunction of a subsuming term and a subsumed one, the subsuming term is deleted, and said to be absorbed in the subsumed one.

A prime implicant (PI) of a switching function S: A term that implies S, such that no other term subsumed by it implies S [30,31].

3. PRELIMINARIES

3.1 Success/ Failure Functions

For a coherent system, the success function $S(X)$ is monotonically increasing, and its complement the failure function $\bar{S}(X)$ is monotonically decreasing [32,33]. Hence S is expressible as a sum of products of the uncomplemented literals X_i alone ($\bar{S}$ is expressible as a sum of products of the complemented literals $\overline{X}_i$ alone) [30,31]. The minimal sum-of-products (s-o-p) expression for success (or failure) is unique since each of the products appearing in it represents an essential or core PI of the function [34,35].

For a noncoherent system, the minimal sum (minimal disjunctive (s-o-p) form) of the success or failure function is the union of certain PIs of the function that may or may not be essential, so that the minimal sum is not usually unique [20,36].

3.2 Capacity and Its Mean

The function $C_{ij}(\boldsymbol{X})$, which is as an expression of the source-to-terminal capacity as a function of element successes is a real valued function of binary arguments. Therefore, the function $C_{ij}(\boldsymbol{X})$ conforms to the rules of the algebraic decomposition relation of a pseudo-Boolean (switching) function [4].

$$\begin{aligned} C_{ij}(X) &= \overline{X}_l\, C_{ij}(X|O_l) + X_l C_{ij}(X|1_l) \\ &= (1 - X_l) C_{ij}(X|0_l) + X_l\, C_{ij}(X|1_l) \\ &= C_{ij}(X|O_l) + [\, C_{ij}(X|1_l) - C_{ij}(X|O_l)] X_l\,,\ l = 1,2,\dots,n \end{aligned} \quad (1)$$

Equation (1) can be validated through proof by perfect induction covering all cases or values of X, viz., $\{X|0_l\}$ and $\{X|1_l\}$. This decomposition relation of $C_{ij}(X)$ can be used to deduce many properties of it as a pseudo-switching function, including, in particular, its being a multi-affine function, and the fact that it can be expressed as a sum-of-products form, where the term 'sum' here refers to its genuine meaning of real addition. Moreover, $C_{ij}(X)$ can be viewed as an assignment of a real number for each of the possible 2^n values of the input vector X [4].

The mean (expected) value of the random function $C_{ij}(X)$, when written in sum-of-products form, equates to;

$$E\{C_{ij}(X)\} = E\{C_{ij}\}(p)\,,$$

and can be directly obtained (on a one-to-one basis) from $C_{ij}(X)$ (s-o-p) by introducing the component means $p_l = \mathrm{E}\{X_l\}$ and $q_l = E\{\overline{X}_l\}$, in place of the corresponding Boolean arguments X_l, and $\overline{X}_l$, namely [4],

$$C_{ij}(\boldsymbol{X})(\text{s-o-p}) \xleftrightarrow{\{X_l, \overline{X}_l\} \leftrightarrow \{p_l, q_l\}} E\{C_{ij}\}(\boldsymbol{p})(\text{s-o-p}) \quad (2)$$

Another subtle replacement that is implicit in (2) pertains to substituting arithmetic multiplication in the R.H.S. for the logical multiplication in the L.H.S., a substitution that is not explicitly apparent since both operations are represented by juxta-positioning. Equation (2) is an immediate result of the condition that the mean of a sum is the sum of means and that the $X_l's$ are statistically independent. It is important to note that the capacity $C_{ij}(X)$ and its square $C_{ij}^2(X)$ are both pseudo-switching functions. Thus, to readily convert $C_{ij}^2(X)$ into its mean, $C_{ij}^2(X)$ can be represented in s-o-p form, namely [4]

$$C_{ij}^2(\boldsymbol{X})(\text{s-o-p}) \xleftrightarrow{\{X_l, \overline{X}_l\} \leftrightarrow \{p_l, q_l\}} E\{C_{ij}^2\}(\boldsymbol{p})(\text{s-o-p}) \quad (3)$$

Equations (2) and (3) show that computing the mean $E\{C_{ij}\}$ and the variance of the capacity

$VAR\{C_{ij}\} = E\{C_{ij}^2\} - \left(E\{C_{ij}\}\right)^2$ can be achieved by ensuring that both the capacity itself and its square are expressed in s-o-p form [4].

3.3 Reliability/Unreliability Functions

Rule 1: A reliability function $R(p)$ is a multiaffine function, and hence can be determined uniquely in terms of 2^n coefficients. In fact, the reliability function $R(p)$ can be written in the form [14]:

$$R(p) = C_0 + \sum_{i=1}^{n} C_i\, p_i + \sum_{1\le i} \sum_{j\le n} C_{ij}\, p_i p_j + \sum_{1\le i} \sum_{j<k} \sum_{k\le n} C_{ijk}\, p_i p_j p_k + \dots + C_{123\dots n}\, p_1 p_2 \cdot\cdot p_n \quad (4)$$

Or more compactly as:

$$R(p) = Z^T(p)\, C \quad (5)$$

Where the vectors $Z(p)$ and C are:

$$Z(p) = [1,\, p_1,\, p_2, \dots,\, p_n,\, p_1 p_2, \dots,\, p_{(n-1)}\, p_n, \dots,\, p_1 p_2 \cdot\cdot p_n]^T \quad (6)$$

$$C = [\, C_0,\, C_1,\, C_2, \dots, C_n,\, C_{12}, \dots,\, C_{(n-1)n}, \dots,\, C_{12..n}]^T \quad (7)$$

The dimension of either $Z(\boldsymbol{p})$ or $\boldsymbol{C}$ is $\left[\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}\right] = 2^n$

4. EXHAUSTIVE TESTS

4.1 Test 1

The symbolic expression of the random function $C_{ij}(X)$ is correct if and only if (iff) it is a multiaffine function that has a correct 'truth table,' i.e., it yields the correct value of C_{ij} for the values of the input vector X implied by all the states of the system, i.e. for the 2^n values X takes when each of its components are allowed to take either one of its two possible values of 0 or 1.This test might alternatively be applied to the expected (mean) value $E\{C_{ij}\}(p)$, which again must be a multiaffine function of a correct 'truth table.' A subtle difference in this case is that p is real-valued (in the unit interval [0.0, 1.0], but we assign to it only the same binary combinations assigned to X above. In both cases of $C_{ij}(X)$ or $E\{C_{ij}\}(p)$, it might be more convenient to construct the 'truth table' in Karnaugh-map form as can be seen in Fig. 2.

1.2 Example 1

The following multiaffine reliability expression is obtained in Rushdi [4] for the small bridge-system shown in Fig. 1, whose branch capacities are: $\boldsymbol{c} = [\,10\ \ 4\ \ 5\ \ 3\ \ 4\,]^T$

The final s-o-p expression for $C_{14}(\boldsymbol{X})$ and its mean are

$$C_{14}(X) = 4X_5\left(X_2 + X_1\overline{X}_2X_3\right) + 3X_4\left(X_1 + \overline{X}_1X_2X_3\overline{X}_5\right) \quad (8)$$

$$E\{C_{14}\}(p) = 4p_5(p_2 + p_1q_2p_3) + 3p_4(p_1 + q_1p_2p_3q_5) \quad (9)$$

If all branches have the same reliability p and unreliability q, then

$$E\{C_{14}\}(p) = 4p^2(1 + qp) + 3p^2(1 + q^2p\,) = p^2\left(7 + qp(4 + 3q)\right) \quad (10)$$

For $p = 0.9$ and $q = 0.1$

$E\{C_{14}\}(p) = 5.98347$

Each of the functions $C_{14}(X_1, X_2, X_3, X_4, X_5)$ and $E\{C_{14}\}(p_1, p_2, p_3, p_4, p_5)$ is a multiaffine function. Table 1 shows that either expression also yields correct results for the 32 values comprising the system states, or the input domain of the 'truth table.'. Expressions (8)-(10) are, therefore, correct.

Test 1 is tedious and impractical even for expressions of moderate size. Table 1 shows that certain patterns of repetitions take place in the checks of Test 1. To save some work, these repetitions can be exploited by replacing each group of similar checks by one.

4.3 Rule 2

If either the expression $C_{ij}(X)$ or $E\{C_{ij}\}(p)$ is functionally correct for a value of X (or p) implied by a prime implicant (PI) P of system success/failure function, then it is correct for the values of X implied by all the single-cell implicants (minterms) M that subsume the prime implicant P.

4.4 Example 2

The s-o-p expression (8) yields the correct result of

$$C_{14}(\boldsymbol{X}) = 4(1)\left(X_2 + (1)\overline{X}_2(1)\right) + 3X_4\left((1) + (0)X_2(1)(0)\right) = 4 + 3X_4 \quad (11)$$

for $X = [1 - 1 - 1]^T$, where $(-)$ denotes a don't-care value. This X corresponds to the term $X_1X_3X_5$ which is a PI of the system success function. This verifies that (8) is correct for the four values of X given by $[10101]^T, [\,11101]^T, [10111]^T, [\,11111]^T$. Correctness of (11) might be verified via the min-cut max-flow theorem, namely

$$C_{14}(X) = \min\{10 + 4X_2, 10 + 5 + 4, 5 + 4X_2 + 3X_4, 4 + 3X_4\} = 4 + 3X_4.$$

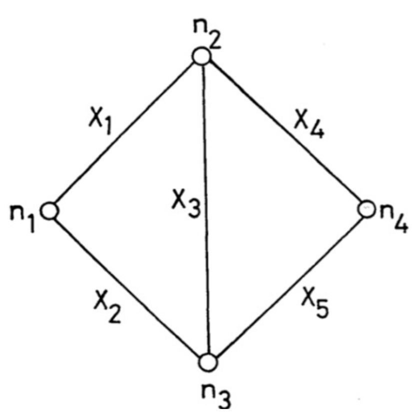

Fig. 1. A 5-branch bridge network of a capacity vector $c = [\,10\ \ 4\ \ 5\ \ 3\ \ 4\,]^T$

Table 1. Exhaustive test for all 32 states of example 1. Each of the capacity function and its expectation function is zero for a failed state, and the two functions share the same non-zero value for a success state

i	X_1	X_2	X_3	X_4	X_5	$C_{14}(X)$	$E\{C_{14}\}(p)$ when $p = X$	Successful states
0.	0	0	0	0	0	0	0	No
1.	1	0	0	0	0	0	0	No
2.	0	1	0	0	0	0	0	No
3.	1	1	0	0	0	0	0	No
4.	0	0	1	0	0	0	0	No
5.	1	0	1	0	0	0	0	No
6.	0	1	1	0	0	0	0	No
7.	1	1	1	0	0	0	0	No
8.	0	0	0	1	0	0	0	No
9.	1	0	0	1	0	3	3	Yes
10.	0	1	0	1	0	0	0	No
11.	1	1	0	1	0	3	3	Yes
12.	0	0	1	1	0	0	0	No
13.	1	0	1	1	0	3	3	Yes
14.	0	1	1	1	0	3	3	Yes
15.	1	1	1	1	0	3	3	Yes
16.	0	0	0	0	1	0	0	No
17.	1	0	0	0	1	0	0	No
18.	0	1	0	0	1	4	4	Yes
19.	1	1	0	0	1	4	4	Yes
20.	0	0	1	0	1	0	0	No
21.	1	0	1	0	1	4	4	Yes
22.	0	1	1	0	1	4	4	Yes
23.	1	1	1	0	1	4	4	Yes
24.	0	0	0	1	1	0	0	No
25.	1	0	0	1	1	3	3	Yes
26.	0	1	0	1	1	4	4	Yes
27.	1	1	0	1	1	7	7	Yes
28.	0	0	1	1	1	0	0	No
29.	1	0	1	1	1	7	7	Yes
30.	0	1	1	1	1	4	4	Yes
31.	1	1	1	1	1	7	7	Yes

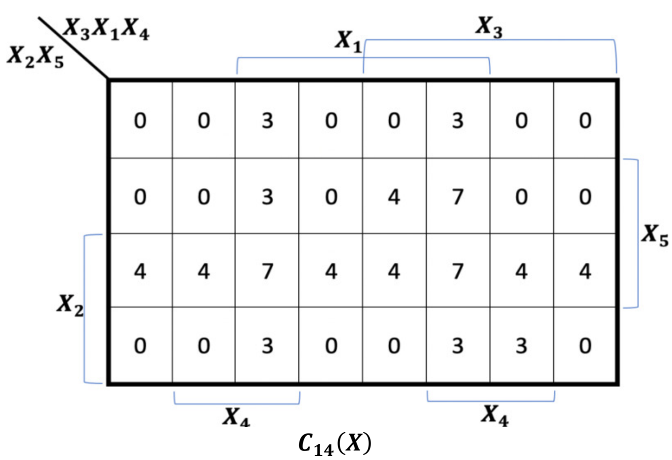

Fig. 2. Modified Karnaugh map representing the pseudo-Boolean function $C_{14}(X)$

With the aid of the success version of rule 2, Test 1 can be partially replaced by a Test 2 that requires less work for success states. To obtain a test that is faster for all states, we now introduce Test 2, which not only benefits of the success version of rule 2, but also utilizes its dual or failure version, which covers failure states collectively by checking the prime implicants (rather than individual cells or configurations) of failure.

4.5 Test 2

The symbolic expression of each of the random capacity pseudo-Boolean function $C_{ij}(X)$ and the random success Boolean function $S_{ij}(X)$ is correct if and only if it is a multiaffine function that has a correct 'truth table,' i.e., it yields the correct results of 0 or 1 of $S_{ij}(X)$ and yields the correct value of C_{ij} for the values of the input vector $\boldsymbol{X}$ implied by:

a) PIs of the system success function that comprise a minimal sum for that function, and
b) PIs of the system failure function that comprise a minimal sum for that function.

As mentioned earlier, sets (*a)* and (*b)* of PIs are unique for coherent systems. For terminal pair reliability set (*a)* contains the minimal *s-t* paths and set (*b)* is that of the minimal *s-t* cutsets, while for overall reliability set (*a)* is the set of all spanning tree terms and set (*b)* is that of the minimal overall cutsets of the network graph [28].

In a capacitated or a flow network, the set (*a)* of *PIs* should contain the minimal valid path groups, which are combinations of the forward *s-t* paths of the network graph, which satisfy the flow constraint, and the set (*b)* should include the minimal valid cut groups, which are subsets of the *s-t* cutsets, which just suffice to prevent the transmission of the required capacity.

Each of the sets (*a)* and (*b)* consists of at most 2^{n-1} PIs [19], and hence Test 2 certainly does not need more checks than Test 1. In fact, the number of checks needed by Test 2 is usually much less than that needed by Test 1. For example, in the case of R_{st} for a coherent system, the number of checks required by Test 2 is of the order of [14]:

$$2^{n-m+2} + 2^{m-2} = 2^n[2^{-m+2} + 2^{-n+m-2}]$$

which is much less than the number 2^n (of checks required by Test 1) when $m \gg 2$ and $n > 2m$. Recall that n and m are the numbers of branches and nodes in the logic diagram of the system.

The conversion from the pseudo-Boolean function $C_{ij}(X)$ to the Boolean function of success $S_{ij}(X)$ can be done by replacing all real non-zero numbers by one and substituting the mathematical operators$\{+, \bullet\}$ by their logic counterparts $\{\vee, \wedge\}$.

4.6 Example 3

Table 2 gives the different checks required by Test 2 to show the correctness of the multiaffine expressions (8)-(10). For convenience, we perform Test 2 for a set of disjoint rather than minimal paths. Fig. 3 shows the Karnaugh map that represents the disjoint paths for the network in Fig. 1, which are employed in Table 2.

Table 2. Test 2 based on the minimal *s-t* cutsets and disjoint paths of example 3

Path	X_1	X_2	X_3	X_4	X_5	$C_{14}(X)$	$S_{14}(X)$	Successful states
X_1X_4	1	-	-	1	-	$3 + X_5(4X_2 + 4\overline{X}_2X_3)$	1	Yes
$\overline{X}_1X_2X_5$	0	1	-	-	1	4	1	Yes
$X_1X_2\overline{X}_4X_5$	1	1	-	0	1	4	1	Yes
$X_1\overline{X}_2X_3\overline{X}_4X_5$	1	0	1	0	1	4	1	Yes
$\overline{X}_1X_2X_3X_4\overline{X}_5$	0	1	1	1	0	3	1	Yes
Cut	X_1	X_2	X_3	X_4	X_5	$C_{14}(X)$	$S_{14}(X)$	**Successful states**
$\overline{X}_1\overline{X}_2$	0	0	-	-	-	0	0	No
$\overline{X}_4\overline{X}_5$	-	-	-	0	0	0	0	No
$\overline{X}_1\overline{X}_3\overline{X}_5$	0	-	0	-	0	0	0	No
$\overline{X}_2\overline{X}_3\overline{X}_4$	-	0	0	0	-	0	0	No

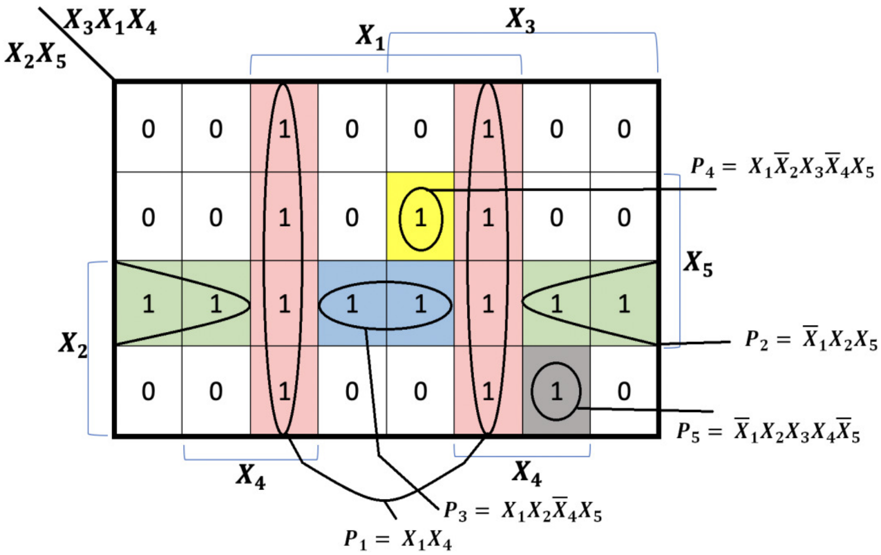

Fig. 3. Modified Karnaugh map representing the disjoint s-t path $S_{14}(X)$ for the network of example 3

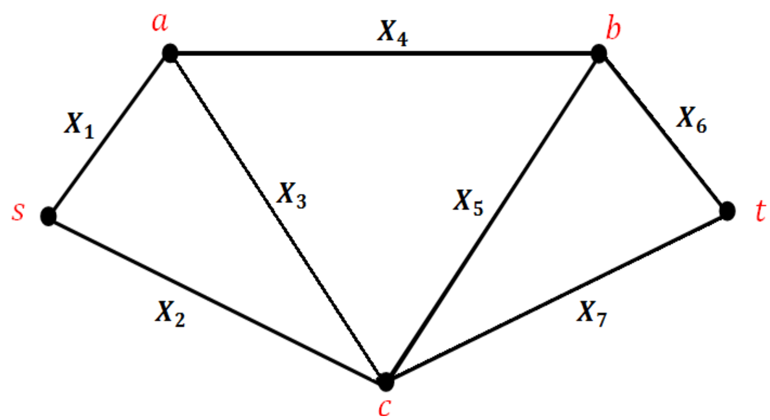

Fig. 4. A 7-branch bridge network of a capacity vector $c = [\,6\ \ 7\ \ 4\ \ 10\ \ 5\ \ 3\ \ 4\,]^T$

4.7 Example 4

The following reliability expression is given in Rushdi and Alsalami [22] for a 7-branch bridge network in Fig. 4, whose branch capacities are: $\boldsymbol{c} = [\,6\ \ 7\ \ 4\ \ 10\ \ 5\ \ 3\ \ 4\,]^T$ .The minimal sum-of-product equation for the pseudo-switching function $C_{st}(X)$ and the corresponding one for its mean are

$$C_{st}(X) = \overline{X}_3\overline{X}_5[\,3\,X_6X_4X_1 + 4\,X_7X_2\,] + \overline{X}_3X_5\,[3\,X_6\left(X_2 + X_1\overline{X}_2X_4(1+X_7)\right) + 4\,X_7(X_2 + X_1\overline{X}_2X_4\overline{X}_6)\,] + X_3\overline{X}_5\,[4\,X_7\left(X_2 + X_1\overline{X}_2(\overline{X}_6 + \overline{X}_4X_6)\right) + 3\,X_4X_6(X_2 + X_1\overline{X}_2(1+X_7))] + X_3X_5\,[3\,X_6\big(X_1 + 3\,X_7X_4\overline{X}_2X_1 + \overline{X}_1X_2\big) + 4\,X_7\big(X_2 + \overline{X}_6\overline{X}_2X_1\big) + X_7X_6\overline{X}_4\overline{X}_2X_1\,] \quad (12a)$$

$$E\{C_{st}\}(p) = q_3q_5[3\,p_6p_4p_1 + 4\,p_7p_2] + q_3p_5[\,3\,p_6(\,p_2 + p_1q_2p_4\,(1+p_7)\,) + 4\,p_7(p_2 + p_1q_2p_4q_6)] + p_3q_5\big[\,4\,p_7\,(\,p_2 + p_1q_2(q_6 + q_4p_6)\big) + 3\,p_4p_6\big(p_2 + p_1q_2(1+p_7)\big)\big] + p_3p_5[\,3\,p_6(p_1 + 3\,p_7p_4q_2p_1 + q_1p_2) + 4\,p_7(p_2 + q_6q_2p_1) + p_7p_6q_4q_2p_1] \quad (12b)$$

The function $E\{C_{st}\}(p_1, p_2, p_3, p_4, p_5, p_6, p_7)$ is multiaffine. Table 3 shows the correctness of the multiaffine expressions (12a) & (12b) by applying Test 2. Fig. 5 shows the representing the disjoint paths for the network in Fig. 4.

Table 3. Test 2 via minimal *s-t* cutsets and disjoint paths of example 4

Path	X_1	X_2	X_3	X_4	X_5	X_6	X_7	$C_{st}(X)$	$S_{st}(X)$	Successful states
X_2X_7	-	1	-	-	-	-	1	$4 + 3X_6\left(X_5 + X_4\overline{X}_5\left(1 + X_1\overline{X}_3\right)\right)$	1	Yes
$X_2X_3X_4\overline{X}_5X_6\overline{X}_7$	-	1	1	1	0	1	0	3	1	Yes
$X_2X_5X_6\overline{X}_7$	-	1	-	-	1	1	0	3	1	Yes
$X_1X_2\overline{X}_3X_4\overline{X}_5X_6\overline{X}_7$	1	1	0	1	0	1	0	3	1	Yes
$X_1\overline{X}_2\overline{X}_3X_4X_6$	1	0	0	1	-	1	-	$3 + 3X_5X_7$	1	Yes
$X_1\overline{X}_2\overline{X}_3X_4X_5\overline{X}_6X_7$	1	0	0	1	1	0	1	4	1	Yes
$X_1\overline{X}_2X_3X_5X_6\overline{X}_7$	1	0	1	-	1	1	0	3	1	Yes
$X_1\overline{X}_2X_3X_7$	1	0	1	-	-	-	1	$4 + 2X_4X_6$	1	Yes
$X_1\overline{X}_2X_3X_4\overline{X}_5X_6\overline{X}_7$	1	0	1	1	0	1	0	3	1	Yes
Cut	X_1	X_2	X_3	X_4	X_5	X_6	X_7	$C_{st}(X)$	$S_{st}(X)$	**Successful states**
$\overline{X}_1\overline{X}_2$	0	0	-	-	-	-	-	0	0	No
$\overline{X}_6\overline{X}_7$	-	-	-	-	-	0	0	0	0	No
$\overline{X}_2\overline{X}_3\overline{X}_4$	-	0	0	0	-	-	-	0	0	No
$\overline{X}_4\overline{X}_5\overline{X}_7$	-	-	-	0	0	-	0	0	0	No
$\overline{X}_1\overline{X}_3\overline{X}_5\overline{X}_7$	0	-	0	-	0	-	0	0	0	No
$\overline{X}_2\overline{X}_3\overline{X}_5\overline{X}_6$	-	0	0	-	0	0	-	0	0	No

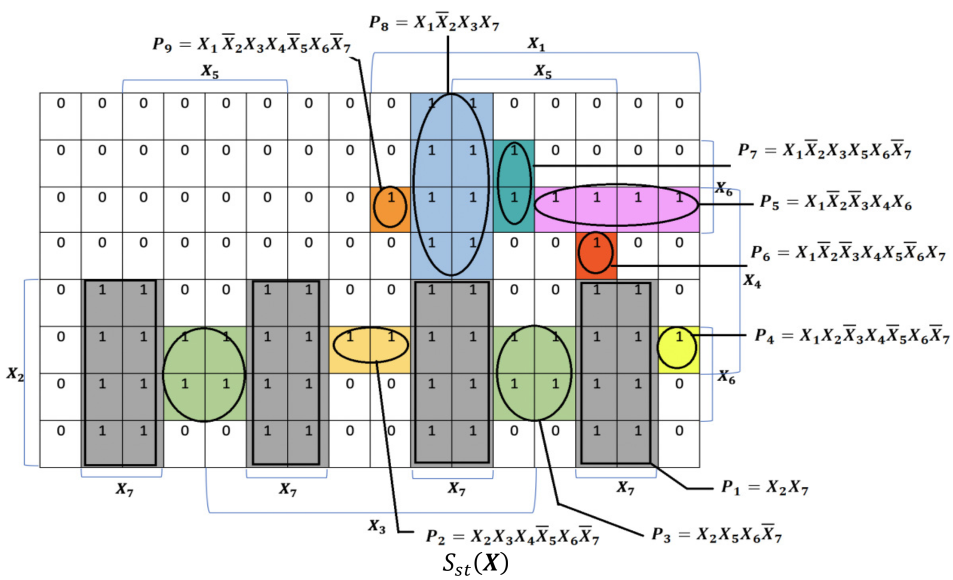

Fig. 5. Modified Karnaugh map representing the disjoint s-t path $S_{st}(X)$ for the network of example 4

4.8 Example 5

The following multiaffine reliability expression is obtained for a 9-branch network in Fig. 6, whose branch capacities are:

$$c = [\,2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\,]^T$$

The minimal sum-of-product equation for the pseudo-switching function $C_{st}(X)$ and the corresponding one for its mean are

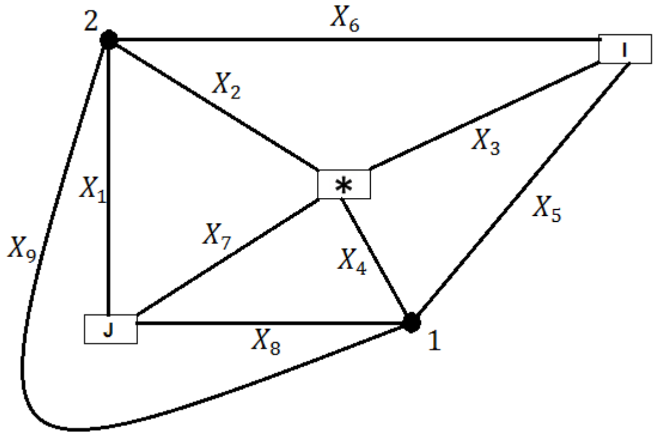

Fig. 6. A 9-branch (corridor) ecological network of a capacity vector $\boldsymbol{c} = [\,2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\,]^T$

$$C_{st}(\boldsymbol{X}) = 11\, X_2X_6X_7 + 10\, X_1X_2\overline{X}_6X_7 + 8(\overline{X}_2X_7 + \overline{X}_1X_2\overline{X}_6X_7) + 5X_4\left(X_5 + \overline{X}_5\left(X_8 + X_6\overline{X}_8X_9(X_1 + \overline{X}_1\overline{X}_2)\right)\right) + 4(X_3 + \overline{X}_1X_2X_4\overline{X}_5X_6\overline{X}_8X_9) + 3X_2\left(X_6\overline{X}_7 + \overline{X}_1\overline{X}_6X_9(X_8 + \overline{X}_4X_5\overline{X}_8)\right) + 2X_1\overline{X}_6(X_2\overline{X}_7 + \overline{X}_2X_4\overline{X}_5\overline{X}_8X_9) + X_2\overline{X}_6X_9\left(X_1X_8 + X_5\overline{X}_8(X_1 + \overline{X}_1X_4)\right) \quad (13)$$

$$E\{C_{st}\}(p) = 11\, p_2p_6p_7 + 10\, p_1p_2q_6p_7 + 8(q_2p_7 + q_1p_2q_6p_7) + 5p_4\left(p_5 + q_5\left(p_8 + p_6q_8p_9(p_1 + q_1q_2)\right)\right) + 4(p_3 + q_1p_2p_4q_5p_6q_8p_9) + 3p_2(p_6q_7 + q_1q_6p_9(p_8 + q_4p_5q_8)) + 2p_1q_6(p_2q_7 + q_2p_4q_5q_8p_9) + p_2q_6p_9(p_1p_8 + p_5q_8(p_1 + q_1p_4)) \quad (14)$$

The function $E\{C_{st}\}(p_1, p_2, p_3, p_4, p_5, p_6, p_7)$ is multiaffine. Table 4 shows the correctness of the multiaffine expressions (13) & (14) by applying Test 2. Fig. 7 shows the Karnaugh map that represents the disjoint paths for the network in Fig. 6.

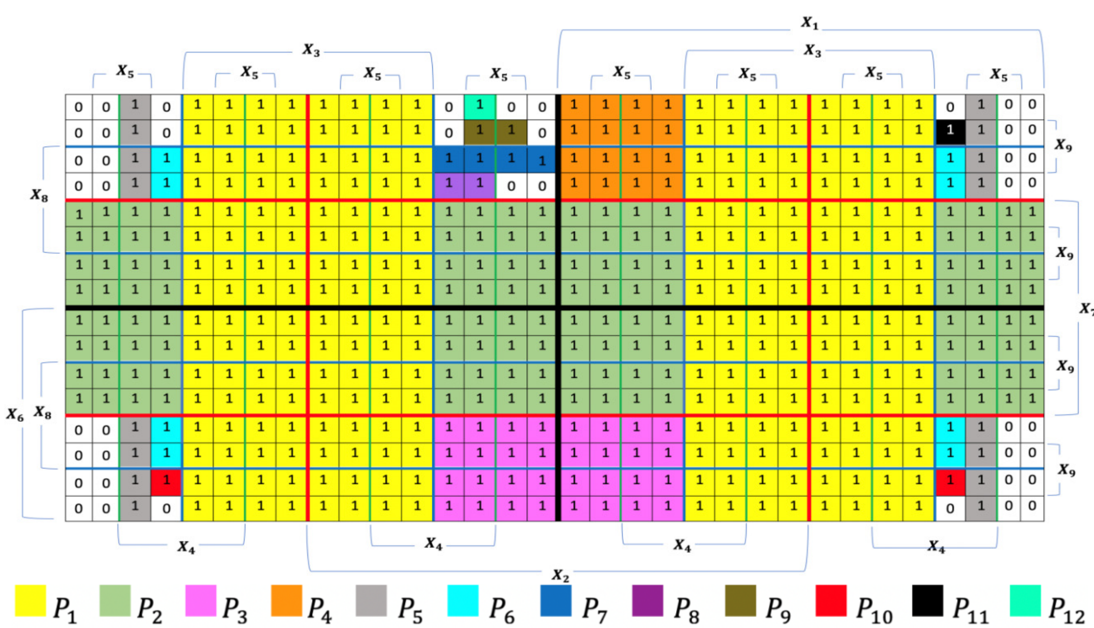

Fig. 7. Modified Karnaugh map representing the disjoint s-t path $S_{st}(X)$ for the network of example 5

Table 4. minimal *s-t* cutsets and disjoint paths test of example 5

Path	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	$E\{C_{st}\}(p)$	$S_{st}(X)$	Successful states
$P_1 = X_3$	-	-	1	-	-	-	-	-	-	(15a)	1	Yes
$P_2 = \overline{X}_3 X_7$	-	-	0	-	-	-	1	-	-	(15b)	1	Yes
$P_3 = X_2\overline{X}_3X_6\overline{X}_7$	-	1	0	-	-	1	0	-	-	(15c)	1	Yes
$P_4 = X_1X_2\overline{X}_3\overline{X}_6\overline{X}_7$	1	1	0	-	-	0	0	-	-	(15d)	1	Yes
$P_5 = \overline{X}_2\overline{X}_3X_4X_5\overline{X}_7$	-	0	0	1	1	-	0	-	-	(15e)	1	Yes
$P_6 = \overline{X}_2\overline{X}_3X_4\overline{X}_5\overline{X}_7X_8$	-	0	0	1	0	-	0	1	-	(15f)	1	Yes
$P_7 = \overline{X}_1X_2\overline{X}_3\overline{X}_6\overline{X}_7X_8X_9$	0	1	0	-	-	0	0	1	1	(15g)	1	Yes
$P_8 = \overline{X}_1X_2\overline{X}_3X_4\overline{X}_6\overline{X}_7X_8\overline{X}_9$	0	1	0	1	-	0	0	1	0	(15h)	1	Yes
$P_9 = \overline{X}_1X_2\overline{X}_3X_5\overline{X}_6\overline{X}_7\overline{X}_8X_9$	0	1	0	-	1	0	0	0	1	(15i)	1	Yes
$P_{10} = \overline{X}_2\overline{X}_3X_4\overline{X}_5X_6\overline{X}_7\overline{X}_8X_9$	-	0	0	1	0	1	0	0	1	(15j)	1	Yes
$P_{11} = X_1\overline{X}_2\overline{X}_3X_4\overline{X}_5\overline{X}_6\overline{X}_7\overline{X}_8X_9$	1	0	0	1	0	0	0	0	1	(15k)	1	Yes
$P_{12} = \overline{X}_1X_2\overline{X}_3X_4X_5\overline{X}_6\overline{X}_7\overline{X}_8\overline{X}_9$	0	1	0	1	1	0	0	0	0	(15l)	1	Yes
Cut	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	$E\{C_{st}\}(p)$	$S_{st}(X)$	**Successful states**
$\overline{X}_1\overline{X}_3\overline{X}_5\overline{X}_6\overline{X}_7\overline{X}_8$	0	-	0	-	0	0	0	0	-	0	0	No
$\overline{X}_2\overline{X}_3\overline{X}_4\overline{X}_7$	-	0	0	0	-	-	0		-	0	0	No
$\overline{X}_2\overline{X}_3\overline{X}_5\overline{X}_7\overline{X}_8\overline{X}_9$	-	0	0	-	0	-	0	0	0	0	0	No
$\overline{X}_1\overline{X}_3\overline{X}_4\overline{X}_6\overline{X}_7\overline{X}_9$	0	-	0	0	-	0	0	-	0	0	0	No

$$E\{C_{st}(\boldsymbol{X}|P_1 = 1)\} = 11\, p_2p_6p_7 + 10\, p_1p_2q_6p_7 + 8p_7(q_2 + q_1p_2q_6) + 5p_4\Big(p_5 + q_5\big(p_8 + p_6q_8p_9(p_1 + q_1q_2)\big)\Big) + 4\,(1 + q_1p_2p_4q_5p_6q_8p_9) + 3p_2\big(p_6q_7 + q_1q_6p_9(p_8 + q_4p_5q_8)\big) + 2p_1q_6(p_2q_7 + q_2p_4q_5q_8p_9) + p_2q_6p_9\big(p_1p_8 + p_5q_8(p_1 + q_1p_4)\big) \tag{15a}$$

$$E\{C_{st}(\boldsymbol{X}|P_2 = 1)\} = 11(p_2p_6) + 10(p_1p_2q_6) + 8(q_2 + q_1p_2q_6) + 5\Big(q_4 + p_4q_5\big(p_8 + p_6q_8p_9(p_1 + q_1q_2)\big)\Big) + 4(q_1p_2p_4q_5p_6q_8p_9) + 3q_1p_2q_6p_9(p_8 + q_4p_5q_8) + 2(p_1q_2p_4q_5q_6q_8p_9) + p_2q_6p_9\Big(p_1(p_8 + p_5q_8(1 + q_1p_4))\Big) \tag{15b}$$

$$E\{C_{st}(\boldsymbol{X}|P_3 = 1)\} = 5\big(q_4p_5 + p_4q_5(p_8 + p_1q_8p_9)\big) + 4(q_1p_4q_5q_8p_9) + 3 \tag{15c}$$

$$E\{C_{st}(\boldsymbol{X}|P_4 = 1)\} = 5p_4(p_5 + q_5p_8) + 2 + p_9(p_8 + p_5q_8) \tag{15d}$$

$$E\{C_{st}(\boldsymbol{X}|P_5 = 1)\} = 5 \tag{15e}$$

$$E\{C_{st}(\boldsymbol{X}|P_6 = 1)\} = 5 \tag{15f}$$

$$E\{C_{st}(\boldsymbol{X}|P_7 = 1)\} = 5p_4 + 3 \tag{15g}$$

$$E\{C_{st}(\boldsymbol{X}|P_8 = 1)\} = 5 \tag{15h}$$

$$E\{C_{st}(\boldsymbol{X}|P_9 = 1)\} = 3(1 + p_4) \tag{15i}$$

$$E\{C_{st}(\boldsymbol{X}|P_{10} = 1)\} = 5 \tag{15j}$$

$$E\{C_{st}(\boldsymbol{X}|P_{11} = 1)\} = 2 \tag{15k}$$

$$E\{C_{st}(\boldsymbol{X}|P_{12} = 1)\} = 5 \tag{15l}$$

4.9 Test 3

The symbolic expression of the random function $C_{ij}(\boldsymbol{X})$ is correct if it is a multiaffine function that reduces to the correct reliability expressions of the subsystems derived from the original system through a Boole-Shannon decomposition [30] with respect to k admissible keystone elements.

Test 3 is useful when the subsystems obtained are simple. If $k = n$, Test 3 and Test 1 are the same.

4.10 Example 6

The problem of Examples 4 for a moderate system having seven s-independent branches is now revisited by applying Test 3.

An application of Test 3, with branches 2, 3, 5 and 6 taken as keystone elements, results in Table 5, which consists of $2^4 = 16$ lines. The results in the 16 lines of this table are correct, as can be easily seen by considering the corresponding subsystems derived from the original system. The table could have been shortened by combining some of its lines, e.g. the entries in lines (1001), (1011), (1101), (1111) are the same, and these lines can be combined as (1 - - 1).

4.11 Example 7

The problem of Examples 5 for a 9-branch network in Fig. 6 is now revisited by applying Test 3. An application of Test 3, with branches 1, 2, 6 and 7 taken as keystone elements, results in Table 6, which consists of $2^4 = 16$ lines. The results in the 16 lines of this table are correct, as can be easily seen by considering the corresponding subsystems derived from the original system.

4.12 Example 8

This example illustrates an alternative way of applying Test 3 to (11) & (12). Initially branch 2 alone is taken as a keystone element. So, decomposing the capacity function $C_{st}(\boldsymbol{X})$ with respect to the indicator variable X_2 in Fig. 4, the following special case of (1) is obtained:

$$C_{st}(\boldsymbol{X}) = X_2 C_{st}(\boldsymbol{X}|1_2) + \overline{X}_2 C_{st}(\boldsymbol{X}|0_2) \tag{16a}$$

The subfunction $C_{st}(X|1_2)$ is the capacity function of the subnetwork derived from the original network and this subnetwork reduces to a series-parallel subsystem as:

$$C_{st}(X|1_2) = \overline{X}_3\overline{X}_5[\,3X_6X_4X_1 + 4X_7] + \overline{X}_3X_5[3X_6 + 4X_7] + X_3\overline{X}_5[4X_7 + 3X_4X_6] + X_3X_5[3X_6 + 4X_7]$$

$$E\{C_{st}(X|1_2)\}(p) = q_3q_5[\,3p_6p_4p_1 + 4p_7] + q_3p_5[3p_6 + 4p_7] + p_3q_5[4p_7 + 3p_4p_6] + p_3p_5[3p_6 + 4p_7] \tag{16b}$$

On the other hand, the subfunction $C_{st}(X|0_2)$ is still complex (does not represent a simple series-parallel system) and consequently it is decomposed further with respect to a second keystone variable, say X_5 to give 2 simple subsystems. i.e.

$$C_{st}(X|0_2) = X_5 C_{st}(X|0_2, 1_5) + \overline{X}_5 C_{st}(X|0_2, 0_5) \tag{16c}$$

Where each of $C_{st}(X|0_2, 1_5)$ and $C_{st}(X|0_2, 0_5)$ represents a simple series-parallel system and therefore, we obtain:

$$C_{st}(X|0_2, 0_5) = X_1\Big[\,3\overline{X}_3X_4X_6 + X_3\Big(4X_1X_7\big(\overline{X}_6 + \overline{X}_4X_6\big) + 3X_4X_6(1 + X_7)\Big)\Big]$$

$$E\{C_{st}(X|0_2, 0_5)\}(p) = p_1\big[\,3q_3p_4p_6 + p_3\big(4p_1p_7(q_6 + q_4p_6) + 3p_4p_6(1 + p_7)\big)\big] \tag{16d.i}$$

Table 5. Test 3 (decomposition test) for all states of example 6

i	X_2	X_3	X_5	X_6	$C_{st}(X)$	$E\{C_{st}\}(p)$
0.	0	0	0	0	0	0
1.	0	0	0	1	$3X_1X_4$	$3p_1p_4$
2.	0	0	1	0	$4X_1X_4X_7$	$4p_1p_4p_7$
3.	0	0	1	1	$3X_1X_4(1 + X_7)$	$3p_1p_4(1 + p_7)$
4.	0	1	0	0	$4X_1X_7$	$4p_1p_7$
5.	0	1	0	1	$3X_1X_4(1 + X_7) + 4X_1\overline{X}_4X_7$	$3p_1p_4(1 + p_7) + 4p_1q_4p_7$
6.	0	1	1	0	$4X_1X_7$	$4p_1p_7$
7.	0	1	1	1	$3X_1(1 + X_4X_7) + X_1\overline{X}_4X_7$	$3p_1(1 + p_4p_7) + p_1q_4p_7$
8.	1	0	0	0	$4X_7$	$4p_7$
9.	1	0	0	1	$4X_7$	$4p_7$
10.	1	0	1	0	$4X_7$	$4p_7$
11.	1	0	1	1	$3 + 4X_7$	$3 + 4p_7$
12.	1	1	0	0	$4X_7$	$4p_7$
13.	1	1	0	1	$4X_7 + 3X_4$	$4p_7 + 3p_4$
14.	1	1	1	0	$4X_7$	$4p_7$
15.	1	1	1	1	$3 + 4X_7$	$3 + 4p_7$

Table 6. Test 3 (decomposition test) for all states of example 7

i	X_1	X_2	X_6	X_7	$C_{st}(X)$	$E\{C_{st}\}(p)$
0.	0	0	0	0	$4X_3 + 5X_4(X_5 + \overline{X}_5X_8)$	$4p_3 + 5p_4(p_5 + q_5p_8)$
1.	0	0	0	1	$8 + 4X_3 + 5X_4(X_5 + \overline{X}_5X_8)$	$8 + 4p_3 + 5p_4(p_5 + q_5p_8)$
2.	0	0	1	0	$4X_3 + 5X_4(X_5 + \overline{X}_5(X_8 + \overline{X}_8X_9))$	$4p_3 + 5p_4(p_5 + q_5(p_8 + q_8p_9))$
3.	0	0	1	1	$8 + 4X_3 + 5X_4\left(X_8 + \overline{X}_8(X_9 + X_5\overline{X}_9)\right)$	$8 + 4p_3 + 5p_4(p_8 + q_8(p_9 + p_5q_9))$
4.	0	1	0	0	$X_4X_5\overline{X}_8X_9 + 3X_9(X_8 + \overline{X}_4X_5\overline{X}_8) + 4X_3 + 5X_4(X_5 + \overline{X}_5X_8)$	$p_4p_5q_8p_9 + 3p_9(p_8 + q_4p_5q_8) + 4p_3 + 5p_4(p_5 + q_5p_8)$
5.	0	1	0	1	$8 + X_4X_5\overline{X}_8X_9 + 3X_9(X_8 + \overline{X}_4X_5\overline{X}_8) + 4X_3 + 5X_4(X_5 + \overline{X}_5X_8)$	$8 + p_4p_5q_8p_9 + 3p_9(p_8 + q_4p_5q_8) + 4p_3 + 5p_4(p_5 + q_5p_8)$
6.	0	1	1	0	$3 + 4(X_3 + X_4\overline{X}_5\overline{X}_8X_9) + 5X_4(X_5 + \overline{X}_5X_8)$	$3 + 4(p_3 + p_4q_5q_8p_9) + 5p_4(p_5 + q_5p_8)$
7.	0	1	1	1	$11 + 4(X_3 + X_4\overline{X}_5\overline{X}_8X_9) + 5X_4(X_5 + \overline{X}_5X_8)$	$11 + 4(p_3 + p_4q_5q_8p_9) + 5p_4(p_5 + q_5p_8)$
8.	1	0	0	0	$2X_4\overline{X}_5\overline{X}_8X_9 + 4X_3 + 5X_4(X_5 + \overline{X}_5X_8)$	$2p_4q_5q_8p_9 + 4p_3 + 5p_4(p_5 + q_5p_8)$
9.	1	0	0	1	$8 + 2X_4\overline{X}_5\overline{X}_8X_9 + 4X_3 + 5X_4(X_5 + \overline{X}_5X_8)$	$8 + 2p_4q_5q_8p_9 + 4p_3 + 5p_4(p_5 + q_5p_8)$
10.	1	0	1	0	$4X_3 + 5X_4\left(X_5 + \overline{X}_5(X_8 + \overline{X}_8X_9)\right)$	$4p_3 + 5p_4(p_5 + q_5(p_8 + q_8p_9))$
11.	1	0	1	1	$8 + 4X_3 + 5X_4\left(X_5 + \overline{X}_5(X_9 + X_8\overline{X}_9)\right)$	$8 + 4p_3 + 5p_4(p_5 + q_5(p_9 + p_8q_9))$
12.	1	1	0	0	$2 + X_9(X_8 + X_5\overline{X}_8) + 4X_3 + 5X_4(X_5 + \overline{X}_5X_8)$	$2 + p_9(p_8 + p_5q_8) + 4p_3 + 5p_4(p_5 + q_5p_8)$
13.	1	1	0	1	$10 + X_9(X_8 + X_5\overline{X}_8) + 4X_3 + 5X_4(X_5 + \overline{X}_5X_8)$	$10 + p_9(p_8 + p_5q_8) + 4p_3 + 5p_4(p_5 + q_5p_8)$
14.	1	1	1	0	$3 + 4X_3 + 5X_4\left(X_5 + \overline{X}_5(X_8 + \overline{X}_8X_9)\right)$	$3 + 4p_3 + 5p_4(p_5 + q_5(p_8 + q_8p_9))$
15.	1	1	1	1	$11 + 4X_3 + 5X_4\left(X_5 + \overline{X}_5(X_9 + X_8\overline{X}_9)\right)$	$11 + 4p_3 + 5p_4(p_5 + q_5(p_9 + p_8q_9))$

$$C_{st}(\boldsymbol{X}|0_2, 1_5) = X_1\overline{X}_3X_4[\,3X_6(1 + X_7) + 4\overline{X}_6X_7] + X_1X_3[\,3X_6(1 + X_4X_7) + X_7(4\overline{X}_6 + \overline{X}_4X_6)]$$

$$E\{C_{st}(\boldsymbol{X}|0_2, 1_5)\}(\boldsymbol{p}) = p_1q_3p_4[\,3p_6(1 + p_7) + 4q_6p_7] + p_1p_3[\,3p_6(1 + p_4p_7) + p_7(4q_6 + q_4p_6)] \qquad (16d.ii)$$

These sub-functions can be substitute into (16a) to get the equivalent expression (11) and its mean (12).

5. FURTHER PROPERTIES OF RELIABILITY EXPRESSIONS

A reliability expression which is obtained by an inclusion-exclusion method [30] takes a form similar to that in (4), i.e. one that does not contain the components unreliabilities $q_i = 1 - p_i$ explicitly. The coefficients $\boldsymbol{C}$ in (4) can take some positive, zero, or negative integral values. For coherent systems $R(p_0) = 0$ and $R(p_{(2^n-1)}) =$

1 where $p_0(p_{(2^n-1)})$ is the vector with all 0(1) components, hence in (2):

$$C_0 = 0; \tag{17}$$

Sum of components of $C = 1$.

Since the unreliability $Q(p)$ is: (18)

$$Q(p) = 1 - R(p) = 1 - Z^T(p)C = Z^T(p)D, \quad 19)$$

Then for a coherent system:

$$D_0 = 1, \tag{20}$$

Sum of components of $D = 0$. (21)

If the reliability expression is obtained by state enumeration, it takes the form:

$$R = M_0 q_1 q_2 q_3 \dots q_n + M_1 p_1 q_2 q_3 \dots q_n + M_2 q_2 p_1 q_3 \dots q_n + M_3 p_1 p_2 q_3 \dots q_n + \dots + M_{(2^n-1)} p_1 p_2 p_3 \dots p_n \tag{22}$$

Where the coefficients M are either 0 or 1. For a coherent system:

$$M_0 = 0 \tag{23}$$

$$M_{(2^n-1)} = 1. \tag{24}$$

If (19) is used to represent Q with the $M's$ replaced by $L's$, then for a coherent system:

$$L_0 = 1 \tag{25}$$

$$L_{(2^n-1)} = 0. \tag{26}$$

It is preferable to obtain reliability expressions by neither of the methods, but by a disjointness method [31]. If a reliability expression for a coherent system is obtained by a disjointness method, then it enjoys the following properties:

1. It is normally in *s-o-p* form; otherwise it can be readily expanded to such a form. Its terms are strictly additive. No numerals appear in it.
2. In any two added terms, there is an element i whose reliability p_i appears in one of the two terms and whose unreliability q_i appears in the other term.
3. R has a single all-p term and no all-q terms, and Q has a single all-q term and no all-p terms.
4. The highest possible cardinality (number of literals) for any term is the number of components in the system.
5. The lowest possible cardinality in any term is:

   a. For R_{st}: the length of the system, i.e. the smallest cardinality (number of components) for a minimal *s-t path*.
   b. For Q_{st}: the width of the system, i.e. the smallest cardinality for a minimal *s-t cutset*.
   c. For R_0: the number of components in a spanning tree, i.e. $(m-1)$.
   d. For Q_0: the smallest degree (number of incident branches) of a node, i.e. the smallest cardinality for a vertex cutset.

It is easy to see that the above properties are enjoyed by (9), (12), (14), (16). If the components reliabilities are allowed to be equal, the reliability function becomes a polynomial of degree n. Properties of these polynomials are discussed in [31] for R_{st}, and in [32] for R_0.

6. THE CASE OF IMPERFECT NODES OF UNLIMITED CAPACITIES

If the m nodes of the network are imperfect, the reliability R becomes a function of $(n+m)$ variables, viz. $R = R(p_1, p_2, \dots, p_n, p_{n1}, p_{n2}, \dots, p_{nm})$. The exhaustive tests of part 4 still apply in this case. However, the following modifications should be noted.

In Test 1: The number of system states increases to 2^{n+m} rather than 2^n.

In Test 2: For R_{st}: a) the number of minimal *s-t* paths remains the same but each path is modifying by multiplying it by the success of the nodes through which it passes, and b) the set of minimal *s-t* cutsets is enlarged to include all feasible *s-t* branch-node cutsets.

For R_0: a) the spanning trees terms are multiplied each by the successes of all the nodes, and b) the set of minimal network cutsets is enlarged to contain the failure of each node.

In Test 3: Some of the nodes may be included beside some admissible branches in the set of keystone elements used in the network decomposition.

Some work can be saved if it is noted that in R_{st} the reliabilities p_{n_s} and p_{n_t} of the input and output nodes appear as multiplicative factors, i.e.

$$R_{st} = p_{n_s}\ p_{n_t}\,[R_{st}]_{\{input\ and\ output\ nodes\ are\ perfect\}} \quad (27)$$

Hence p_{n_s} and p_{n_t} can be factored out and removed from further consideration. More saving is accomplished in testing R_0 since the reliabilities of the different nodes appear as multiplicative factor, i.e.

$$R_0 = p_{n_1}\ p_{n_2} \cdots p_{n_m}\,[R_0]_{\{all\ nodes\ are\ perfect\}} \quad (28)$$

No significant saving is obtained if Q_{st} or Q_0 expressions are handled.

6.1 Example 9

If both nodes and branches of the system in Fig. 1 are imperfect, the terminal-pair reliability is [37]:

$$R = R_{14} = p_{n_1} p_{n_4}\big(p_1 p_{n_2} p_4 + p_2 p_{n_3} p_5 + p_1 p_{n_2} p_3 p_{n_3} p_5 + p_2 p_{n_3} p_3 p_{n_2} p_4 - p_1 p_2 p_{n_2} p_{n_3} p_4 p_5 - p_1 p_{n_2} p_{n_3} p_3 p_4 p_5 - p_1 p_2 p_{n_2} p_{n_3} p_3 p_4 - p_1 p_2 p_{n_2} p_{n_3} p_3 p_5 - p_2 p_{n_2} p_{n_3} p_3 p_4 p_5 + 2 p_1 p_2 p_{n_2} p_{n_3} p_3 p_4 p_5\big). \quad (29)$$

This expression is multiaffine in its 9 variables. To prove its correctness, 512 checks are needed by Test 1, but only 15 (or 13) checks are needed by Test 2. The checks of Test 2 are to see that $R = 1$ for the paths $N_1X_1N_2X_4N_4, N_1X_2N_3X_5N_4, N_1X_1N_2X_3\ N_3X_5N_4, N_1X_2N_3X_3N_2X_4N_4,$ and that $R = 0$ for the cutsets $\overline{N}_1, \overline{X}_1\overline{X}_2, \overline{N}_2\overline{X}_2, \overline{N}_3\overline{X}_1, \overline{N}_2\overline{N}_3, \overline{X}_4\overline{X}_5, \overline{X}_1\overline{X}_3\overline{X}_5, \overline{X}_2\overline{X}_3\overline{X}_4, \overline{N}_2\overline{X}_5, \overline{N}_3\overline{X}_4, \overline{N}_4.$

Alternatively, the factor $p_{n_1}\ p_{n_4}$ can be dropped from (29) and the reminder of R checked to be 1 for $X_1N_2X_4, X_2N_3X_5, X_1N_2X_3N_3X_5, X_2N_3X_3N_2X_4$ and to be 0 for the previous set of cutsets excluding $\overline{N}_1$ and $\overline{N}_4$.

7. CONCLUSIONS

This paper presents three exhaustive tests to show the correctness of reliability expressions in flow networks. The tests apply to reliability as well as unreliability expressions for coherent and noncoherent systems. All the tests and other t results are proved and illustrated by examples. The problem of proving the correctness of a reliability expression seems to have a complexity of the same order as that of the problem of reliability evaluation itself. This means that the exhaustive tests become prohibitively time-consuming for large systems, even when the computer is used to implement them. Therefore, a reliability analyst may feel content, from a practical point of view, just to check the general properties of the expression at hand, and to try a few sample checks which constitute only a part of one of the tests.

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COMPETING INTERESTS

The authors have declared that no competing interests exist.

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APPENDIX: THE PROOFS OF RULES AND TESTS

Proof of rule 1: The Total Probability Theorem is used to expand the reliability function $R(p_1, \dots p_{i-1}, p_i, p_{i+1}, \dots, p_n)$ about its *i-th* variable [14].

$$R(p_1, \dots p_{i-1}, p_i, p_{i+1}, \dots, p_n) = (1-p_i)R(p_1, \dots, p_{i-1}, 0, p_{i+1}, \dots, p_n) + p_i R(p_1, \dots, p_{i-1}, 1, p_{i+1}, \dots, p_n) = R(p_1, \dots, p_{i-1}, 0, p_{i+1}, \dots, p_n) + p_i[\, R(p_1, \dots, p_{i-1}, 1, p_{i+1}, \dots, p_n) - R(p_1, \dots, p_{i-1}, 0, p_{i+1}, \dots, p_n)] \quad (30)$$

Equation (30) means that $R(p)$ is a first-degree polynomial in p_i where $i = 0,1, \dots, n.$ Now to prove that $R(p)$ is completely specified by 2^n coefficients, mathematical induction is used [14]. The result is true for $n = 1$ since $R(p_1)$ is a first-degree polynomial in p_1 , i.e. $R(p_1) = C_0 + C_1 p_1$ and hence is specified by 2^1 coefficients. If the result is assumed true for $n = j - 1$ then it is also true for $n = j$. This is because the function $R(p_1, p_2, \dots, p_j)$ of j variables is [14]:

$$R(p_1, p_2, \dots, p_{j-1}, p_j) = (1 - p_j)R(p_1, p_2, \dots, p_{j-1}, 0) + p_j R(p_1, p_2, \dots, p_{j-1}, 1). \quad (31)$$

The function $R(p_1, p_2, \dots, p_{j-1}, 0)$ and $R(p_1, p_2, \dots, p_{j-1}, 1)$ of $(j-1)$ variables are specified by 2^{j-1} coefficients each, hence from (31) the function $R(p_1, p_2, \dots, p_{j-1}, p_j)$ of j variables is specified by $(2^{j-1} + 2^{j-1}) = 2^j$coefficients. Hence the result is true for any n. *QED*

Proof of Test 1: By rule 1, $R(p)$ must be multiaffine, and hence it can be given by the general form (5). If the input vector p is allowed to take the values $p = p_i$ where $i = 0,1, \dots, (2^n - 1)$ which are all the possible values it can take when its n components are restricted to the values 0 or 1, the 2^n dimensional vector $Z(p)$ takes 2^n linearly independent values. If the reliability expression $R(p)$ satisfies the 2^nlinearly independent equations:

$$R(p_i) = Z^T(p)\, C \qquad i = 0,1, \dots, (2^n - 1)$$

which can be combined into a single matrix equation of the form [14]

$$R = \begin{bmatrix} R(p_0) \\ R(p_1) \\ \dots \\ R(p_{(2^n-1)}) \end{bmatrix} = \begin{bmatrix} Z^T(p_0) \\ Z^T(p_1) \\ \dots \\ Z^T(p_{(2^n-1)}) \end{bmatrix} \quad C = [Z]C \quad (32)$$

Then the matrix $[Z]$ has full rank since its rows $Z^T(p_i)$ are linearly independent, and hence can be inverted to yield the unique value $= [Z]^{-1}R$. Therefore, under the conditions stated in the test, the reliability expression is determined uniquely [14]. *QED*

Proof of rule 2: For each *M* such that *M* subsumes $P{:}\, M = 1$ implies $P = 1$, but $P = 1$ implies R is correct, hence $M = 1$ implies R is correct. *QED*

Proof of Test 2: According to rule 2, set *a* guarantees that R yields correct results for all success states of the system, while set *b* guarantees its correctness for all failure states. Hence R yields correct results for all states of the system, and since it is multiaffine, then according to test 1, it is correct. No member of the sets *a* or *b* is redundant; for each member there is at least one state covered by that member alone. Hence the amount of work needed by test 2 is minimal. *QED*

Proof of Test 3: Without loss of generality, the keystone variables can be considered as the k variables in $\boldsymbol{p}$ that appear first, hence by successive applications of total probability theorem, the reliability $R(p)$ is [14]:

$$R(p) = q_1 q_2 \ldots q_k \, R(0,0,\ldots,0,p_{k+1},\ldots,p_n) + p_1 q_2 \ldots q_k \, R(1,0,\ldots,0,p_{k+1},\ldots,p_n) + \cdots + p_1 p_2 \ldots p_k R(1,1,\ldots,1,p_{k+1},\ldots p_n) \tag{33}$$

Therefore, if $R(p)$ is multiaffine in the keystone variables and the conditional reliabilities $R(0,0,\ldots,p_{k+1},\ldots,p_n),\ldots,etc.$ are correct, then $R(p)$ is correct, and vice versa. *QED*

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