Journal of Advances in Mathematics and Computer Science

28(5): 1-7, 2018; Article no.JAMCS.43076

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Application of the Euler Sequence in Continuous Compounding

Okorie, Kalu Okam[1*], Jamiu, Garba[2], A. M. Mamu[3], Bello, Ridwan Oluwaseun[4] and Suleiman, Saminu Kanti[5]

[1] *Department of Mathematics and Computer Science, Ebonyi State University, Abakaliki, Nigeria.*

[2] *Department of Mathematics, Federal University of Technology, Minna, Nigeria.*

[3] *Department of Statistics, Federal University of Technology, Minna, Nigeria.*

[4] *Department of Mathematical and Computer Sciences, Fountain University, Osogbo, Nigeria.*

[5] *Department of Business Administration, Ecole Superieure Sainte Felicite, Cotonou, Benin.*

Authors' contributions

This work was carried out in collaboration between all authors. Author OKO designed the study, performed the mathematical analysis, wrote the protocol and the first draft of the manuscript. Authors JG and AMM managed the literature searches and provided examples of continuous compounding problems. Authors BRO and SSK managed continuous compounding and gave insightful comments on all sections of the draft manuscript. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2018/43076

Editor(s):

(1) Dr. H. M. Srivastava, Professor, Department of Mathematics and Statistics, University of Victoria, Canada.

(2) Dr. Sheng Zhang, Professor, Department of Mathematics, Bohai University, Jinzhou, China.

Reviewers:

(1) Danjuma Samaila, Adamawa State University Mubi, Nigeria.

(2) W. Obeng-Denteh, Kwame Nkrumah University of Science and Technology, Ghana.

(3) Peter O. Arawomo, University of Ibadan, Nigeria.

(4) Kritsana Sokhuma, Phranakhon Rajabhat University,Thailand.

Complete Peer review History: http://www.sciencedomain.org/review-history/26157

Received: 30 May 2018

Accepted: 01 August 2018

Published: 08 September 2018

Original Research Article

Abstract

In this paper, the Euler number **e**, the Euler sequence and its application in real life situations particularly in business world (financial investment) are discussed. A basic theorem concerning convergence of the Euler sequence in which some lemmas were considered in order to achieve the proof of the theorem is presented.

**Corresponding author: okaco9585@gmail.com*

The paper showed the application of the Euler sequence in continuous compounding and the development of a new formula (INTEREST FORMULA) in continuous compounding. To achieve this, the convergence of the Euler sequence to the Euler number **e** was established and finally, its application in continuous compounding and some striking examples are shown.

Keywords: The Euler number; the Euler sequence; continuous compounding.

1 Introduction

The first reference to the constant, the Euler number, were first published in 1618 in the table of an appendix of a work on logarithms by John Napier. However, this did not contain the constant itself, but simply a list of logarithms calculated from the constant. It is assumed that the table was written by William Oughtred. The discovery of the constant itself is credited to Jacob Bernoulli who in 1683 attempted to find the value of the expression (which is in fact **e**), $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ when he was examining continuous compound interest. The first known use of the constant, represented by **b**, was in correspondence from Gottfried Leibniz to Christiaan Huygens in 1690 to 1691. Leonhard Euler introduced the letter **e** as the base for natural logarithms, writing in a letter to Christian Goldbach on 25 November 1731. He made various discoveries regarding **e** in the following years, but it was not until 1748 when Euler published *introductio in analysin infinitorum* that he gave a full treatment of the ideas surrounding **e**. He showed that $\mathbf{e} = \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}\cdots$ [1] and that **e** is the limit of $\left(1+\frac{1}{n}\right)^n$ as n tends to infinity.

The application of the Euler sequence $\left(1+\frac{1}{n}\right)^n$ in continuous compounding is of interest in this paper, considering the basic theorem and some lemmas in mathematical analysis in its accomplishment.

2 The Euler Sequence

Definition 1. *A sequence* $\{x_n\}_{n=k}^{\infty}$ *of the form* $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=k}^{\infty}$ *where* $k\in\mathbb{N}$*, is called the Euler sequence.*

Theorem 1. *The Euler sequence* $x_n := \left(1+\frac{1}{n}\right)^n$ *converges to* **e**; *where*

$$\mathbf{e} := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Lemma 1. *Let* $n\in\mathbb{N}\cup\{0\}$, *then* $\frac{1}{n!}\le\frac{1}{2^{n-1}}$.

Proof: The proof of the above lemma by mathematical induction is thus:

$$\begin{aligned}\text{Let }\ p(n) &:= \frac{1}{n!}\le\frac{1}{2^{n-1}}\\ \text{then }\ p(0) &\implies \frac{1}{0!}\le\frac{1}{2^{-1}}\\ &\implies 1\le 2.\ \text{Therefore, } p(0) \text{ is true.}\end{aligned}$$

Assume that $p(k)$ is true for some $k\in\mathbb{N}$. That is $\dfrac{1}{k!}\le\dfrac{1}{2^{k-1}}$.

Next, showing that $p(k+1)$ is also true $\forall\ k \in \mathbb{N}$.

$$\frac{1}{(k+1)!} = \frac{1}{(k+1)k!} = \frac{1}{(k+1)} \cdot \frac{1}{k!} \leq \frac{1}{(k+1)} \cdot \frac{1}{2^{k-1}}$$

Observe that $k+1 \geq 2$ for all k $\in \mathbb{N}$.

$$\text{But } k+1 \geq 2 \implies \frac{1}{k+1} \leq \frac{1}{2}.$$

$$\text{So that } \frac{1}{(k+1)} \cdot \frac{1}{2^{k-1}} \leq \frac{1}{2} \cdot \frac{1}{2^{k-1}} = \frac{1}{2} \cdot \frac{2}{2^k} = \frac{1}{2^k}$$

$$\implies \frac{1}{(k+1)!} \leq \frac{1}{2^k}.$$

Therefore, the inequality is true for all integers $n \in \mathbb{N} \cup \{\,0\,\}$.

Lemma 2. ***(Bernoulli's inequality)*** *Let $1+p>0$, $p \neq 0$. Then for every integer $n \geq 2$, we have $(1+p)^n > 1+np$.*

Proof: Proof of the above lemma by mathematical induction.

$$\text{Let } p(n) := (\,1+p\,)^n > 1+np.$$

then $p(2) \implies (\,1+p\,)^2 = 1+2p+p^2 > 1+2p$, since $p^2 > 0\ \forall$ p $\in (\,-1\,,\,\infty\,) \setminus \{\,0\,\}$.

Therefore, $p(2)$ is true. Assume that $p(k)$ is true for some $k \in \mathbb{N}$.

That is $(\,1+p\,)^k > 1+kp$ for all $p \in (\,-1\,,\,\infty\,) \setminus \{\,0\,\}$.

Showing that $p(k+1)$ is also true $\forall\ k \in \mathbb{N}$ and for all $p \in (\,-1\,,\,\infty\,) \setminus \{\,0\,\}$.

$$(\,1+p\,)^{k+1} = (\,1+p\,)^k(\,1+p\,) > (\,1+kp\,)(\,1+p\,)$$

but, $(\,1+kp\,)(\,1+p\,) = 1+(\,k+1\,)p+kp^2 > 1+(\,k+1\,)p$, since $kp^2 > 0$

$$\implies (\,1+p\,)^{k+1} > 1+(\,k+1\,)p$$

Therefore, the inequality is true for all integers $n \geq 2$.

See also Chidume C. E. and Chidume C. O. [2]

At this point, the proof of Theorem 1 is produced.

Proof: First, proof that $x_n := (\,1+\frac{1}{n}\,)^n$ is bounded. By Binomial expansion,

$$\left(1+\frac{1}{n}\right)^n = 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3}+\cdots+\frac{n(n-1)\cdots 3\cdot 2\cdot 1}{n!}\cdot\frac{1}{n^n}$$

$$=2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

With all the brackets on the right hand side being positive numbers, then:

$$2<2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

and

$2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) < 2+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$

$\implies 2 < (\,1+\frac{1}{n}\,)^n < 2+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$

but,

$$2+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}<2+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}} \qquad \text{By Lemma 1}$$

$$2+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}=1+\underbrace{1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}}_{S_n}$$

S_n is a geometric series, hence $S_n = \frac{1(1-(\frac{1}{2})^n)}{(1-\frac{1}{2})} = 2(1-(\frac{1}{2})^n) < 2 \quad \forall\ n \in \mathbb{N}$. Therefore, $2+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} < 1+2=3$. Hence, $2 < (\ 1+\frac{1}{n}\ )^n < 3$. Which shows that $(\ 1+\frac{1}{n}\ )^n$ is bounded.

See also Malik S. C. and Savita A. [3]

Next, is to show that $(\ 1+\frac{1}{n}\ )^n$ is monotone. Since

$$x_n = \left(\ 1+\frac{1}{n}\ \right)^n \qquad take \qquad x_{n-1} = \left(\ 1+\frac{1}{n-1}\ \right)^{n-1}$$

So that

$$\frac{x_n}{x_{n-1}} = \left(\frac{n^2-1}{n^2}\right)^n\left(\frac{n}{n-1}\right) = \left(1-\frac{1}{n^2}\right)^n\left(\frac{n}{n-1}\right) \qquad n \geq 2$$

$$\left(1-\frac{1}{n^2}\right)^n\left(\frac{n}{n-1}\right) > \left(1-\frac{1}{n}\right)\left(\frac{n}{n-1}\right) \qquad \text{(By Bernoulli inequality)}$$

$$\left(1-\frac{1}{n}\right)\left(\frac{n}{n-1}\right) = 1$$

$$\implies \frac{x_n}{x_{n-1}} > 1$$

$$\implies x_n > x_{n-1}\ \forall\ n \geq 2$$

See also Chidume C. E. and Chidume C. O. [2]

Therefore x_n is monotone increasing. Since x_n is bounded and monotone, it converges.

Next shows that the limit of x_n, $n \to \infty$ is the Euler number $\mathbf{e}$. Recall, that it was earlier shown that $2 < (\ 1+\frac{1}{n}\ )^n < 3$. Also note that

$$\left(\ 1+\frac{1}{n}\ \right)^n = 2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

$\implies 2 < 2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) < 3$. Now, by taking the limits of all sides of the last inequality to get:
$\lim_{n\to\infty} 2 < \lim_{n\to\infty}\left(2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)\right) <$ $\lim_{n\to\infty} 3 \implies 2 < 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots < 3$. But, $\mathbf{e^x} = \frac{1}{0!}+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}\cdots$ [4]. Take $x=1$ therefore $\mathbf{e^1} = \mathbf{e} = \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}\cdots$ but, $2 < \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}\cdots < 3$ $\implies 2 < \mathbf{e} < 3$. Hence, $\lim_{n\to\infty}(\ 1+\frac{1}{n}\ )^n = \mathbf{e}$.

3 Continuous Compounding

This section discusses the application of the Euler sequence in continuous compounding. Continuous compounding is a very crucial concept in financial mathematics and its application in business world (Financial investment) cannot be underestimated. In financial mathematics, $FV = PV\left(1+\frac{r}{n}\right)^n$

as the formula for compound interest compounded "n" times in a year. Therefore for "t" years, $FV = PV\left(1+\frac{r}{n}\right)^{nt}$, where FV = Future Value, PV = Present Value, r = Annual interest rate, t = Number of years, n = Frequency of compounding.

See also Campbell S. [5]

It is important to note that the more frequent an investment is compounded, the greater the return it will produce. However, there is a limit on the interest accumulation ($\lim(FV - PV)$), irrespective of how frequent the compounding is ($n \to \infty$). Therefore,

$$\lim_{n\to\infty}(FV-PV) = \lim_{n\to\infty}\left(PV\left(1+\frac{r}{n}\right)^{nt} - PV\right) = \lim_{n\to\infty}\left(PV\left(\left(1+\frac{r}{n}\right)^{nt}-1\right)\right) \tag{3.1}$$

$$= PV\lim_{n\to\infty}\left(\left(1+\frac{r}{n}\right)^{nt}-1\right) = PV\left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} - \lim_{n\to\infty}1\right) \tag{3.2}$$

but, $\lim_{n\to\infty} 1 = 1$ and $\lim_{n\to\infty} PV = PV$ since PV is a constant.

Estimating the $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}$. Take $n = kr$, where $k \in \mathbb{N}$

and r is a constant. So that $n \to \infty \implies kr \to \infty \implies k \to \infty$ since r is a constant.

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = \lim_{k\to\infty}\left(1+\frac{r}{kr}\right)^{krt} = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{krt}$$

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{krt} = \left(\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k}\right)^{rt} = \mathbf{e}^{\mathbf{rt}} \qquad \text{(By theorem 1)}$$

From Equation 3.1, Equation 3.2 and Theorem 1,

$$\lim_{n\to\infty}(FV-PV) = PV\left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} - \lim_{n\to\infty}1\right)$$
$$= PV\left(\mathbf{e}^{\mathbf{rt}}-1\right)$$

But, $FV - PV$ = INTEREST and INTEREST is a constant, therefore,

$$\lim_{n\to\infty}(FV-PV) = PV\left(\mathbf{e}^{\mathbf{rt}}-1\right) \tag{3.3}$$

$$\implies \lim_{n\to\infty}(INTEREST) = PV\left(\mathbf{e}^{\mathbf{rt}}-1\right) \tag{3.4}$$

$$\implies INTEREST = PV\left(\mathbf{e}^{\mathbf{rt}}-1\right) \qquad \text{(INTEREST FORMULA)} \tag{3.5}$$

$$\implies FV - PV = PV\mathbf{e}^{\mathbf{rt}} - PV \tag{3.6}$$

$$\implies FV = PV\mathbf{e}^{\mathbf{rt}} \tag{3.7}$$

This is the formula for continuous compounding often used in business world (Financial investment) today.

It should be noted that in the process of showing the formula for continuous compounding, a very important formula is produced, and that is:

$$INTEREST = PV\left(\mathbf{e}^{\mathbf{rt}}-1\right) \qquad \text{(INTEREST FORMULA)}$$

4 Examples

This section considers some examples on continuous compounding discussed in the previous section.

Example 1: In a financial investment, the amount accumulated after "t" years is directly proportional to the investment made. In this arrangement, if an investment of \$20,000 accumulates to \$45,000 after 5 years calculate:

a The rate at which the investment is made;

b The future value of the investment made after 10 years; and

c The interest earned from the investment after 15 years.

Solution: Let P_t be the accumulated investment after "t" years and at $t = 0$ take P_0 as the initial investment. So that:

a From the question, $\frac{dP_t}{dt}\ \alpha\ P_t$. But,

$$\frac{dP_t}{dt}\ \alpha\ P_t \implies \frac{dP_t}{dt} = rP_t \qquad \text{where r is a constant} \tag{4.1}$$

$$\implies \frac{dP_t}{P_t} = rdt \tag{4.2}$$

$$\text{Integrating both sides:} \int \frac{dP_t}{P_t} = \int rdt \tag{4.3}$$

$$\implies \log_e P_t = rt + k \tag{4.4}$$

$$\implies P_t = \mathbf{e^{rt+k}} \tag{4.5}$$

$$= \mathbf{e^{rt}} \cdot \mathbf{e^{k}} \qquad \text{take } \mathbf{e^k} = A \tag{4.6}$$

$$\text{Hence, } P_t = A \cdot \mathbf{e^{rt}} \tag{4.7}$$

See also [6]
At $t = 0$, $P_0 = A \cdot \mathbf{e^0} = A \implies A = P_0$. Therefore $P_t = P_0 \cdot \mathbf{e^{rt}}$. Comparing $P_t = P_0 \cdot \mathbf{e^{rt}}$ with the continuous compounding formula, that is $FV = PV\mathbf{e^{rt}}$ shows that $P_t = FV$ and $P_0 = PV$; and therefore, the arrangement is a continuous compounding arrangement. Hence $FV = PV\mathbf{e^{rt}}$ will be used in the computation.

Note that if not for the continuous compounding formula, the arrangement used in the investment would not have known.

From the question above, the present Value (PV) = \$20,000, Future Value (FV) = \$45,000, Time (t) = 5 years.

Using $FV = PV\mathbf{e^{rt}}$, therefore,

$$\$45,000 = \$20,000\mathbf{e^{5r}}$$
$$\implies 2.25 = \mathbf{e^{5r}}$$
$$\text{so that } \log_{\mathbf{e}} 2.25 = 5r$$
$$\implies r = \frac{\log_{\mathbf{e}} 2.25}{5}$$
$$\text{Hence, } r = 0.1622\ (approx.) = 16.22\%$$

b $FV = \$20,000 \cdot \mathbf{e}^{\mathbf{10\times 0.1622}} = \$101,264.13$ (approx.)

c Using the interest formula, that is $INTEREST = PV(\mathbf{e}^{\mathbf{rt}} - 1)$, $INTEREST = \$20,000(\mathbf{e}^{\mathbf{15\times 0.1622}} - 1) = \$207,860.20$ (approx.)

Example 2: Mr. Thomas deposited $\$100,000$ compounded at 5% continuously for 5 years. How much is his total interest at the end of the term?

Solution: Two different approaches are used here in order to investigate the easier method.

First method $FV = PV\mathbf{e}^{\mathbf{rt}} = \$100,000 \times \mathbf{e}^{\mathbf{5\times 0.05}} = \$128,402.54$ (approx.), but, $INTEREST = \$128,402.54 - \$100,000 = \$28,402.54$

Second method Using $INTEREST = PV(\mathbf{e}^{\mathbf{rt}} - 1) \implies INTEREST = \$100,000(\mathbf{e}^{\mathbf{5\times 0.05}} - 1) = \$28,402.54$ (approx.). Observe that using interest formula (second method) is better.

5 Conclusion

The authors showed the crucial roles which the Euler number and the Euler sequence played in continuous compounding, and also were able to come up with a formula (INTEREST FORMULA) which can be use in computing interest easily in continuous compounding problems considering present value, rate and time. Some striking examples with regards to continuous compounding were presented and solved.

Competing Interests

Authors have declared that no competing interests exist..

References

[1] Smale S, Hirsch MW, Devaney RL. Differential equations, dynamical systems and an introduction to chaos, Academic Press; 2013.

[2] Chidume CE, Chidume CO. Foundation of mathematical analysis. Ibadan: University Press; 2014

[3] Malik SC, Savita A. Mathematical analysis. Delhi: New Age International limited publishers; 2012

[4] Stroud KA, Dexter j. Booth. Engineering mathematics. New York: Palgrave Macmillan limited; 2007

[5] Campbell S. Financial Institution, Markets and Economics Activity. New York: McGraw-Hill company; 1982

[6] Osisiogu UA. A First Course In Mathematical Analysis And Differential Equation. Festac Town: Bestsoft Educational Books; 2013

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sciencedomain.org/review-history/26157