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Multiple Solutions of Riemann Type of Functional Equations

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Authors' contributions

This work was carried out in collaboration between all authors. Author DG designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript and managed literature searches. Authors TCH and FAM, managed the analyses of the study and literature searches. All authors read and approved the final manuscript.

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Abstract

Linearly independent Dirichlet L-functions satisfying the same Riemann type of functional equation have been supposed for a long time to possess off critical line non trivial zeros. We are taking a closser look into this problem and into its connection with the Generalized Riemann Hypothesis.

Keywords: Riemann type of functional equation; non trivial zeros; Riemann Hypothesis.

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1 Introduction

Starting with the 1935 paper of Potter and Titchmarsh [1], the attempts to find counterexamples to the Generalized Riemann Hypothesis (GRH) multiplied (see [2], [3], [4]). The idea was to use linear combinations of Dirichlet L-series. These are no more Euler products, but their analytic continuations to the whole complex plane can still satisfy some Riemann type of functional equations. There were two approaches to achieve this goal: one was to chose conveniently the coefficients of the respective linear combination, as in the case of the so called Davenport and Heilbronn function [1], [4] and the other one was to use L-functions satisfying the same functional equation, for which any linear combination would do. An exhaustive presentation of this topic can be found in [5]. Important contributions were brought by Voronin [6], Bombieri and Heijhal [7], Bombieri and Mueller [8], Lee [9] etc.

Potter and Titchmarsh thought they had identified for the Davenport and Heilbronn function two such zeros, yet they acknowledged that "the calculations are very cumbrous, and can hardly be considered conclusive". However, more such zeros were indicated in [2], [3] and [4]. A mismatch of Dirichlet characters in the formula for that function brought us to think that (see [10]) the approximation errors which naturally affected their computation produced false off critical line zeros. Our geometric function theory approach in [10] was showing that such zeros cannot exist. We discovered later our mistake and came in [11] with a correction.

In this paper we shall deal with the second approach, namely that of linearly independent L-functions satisfying the same functional equation. We perceive this class of functions not as a source of counterexamples to the GRH, but rather as a confirmation of the theory developed in [12] and [13] and [14] in which we have shown that if a general Dirichlet series can be continued analytically to the whole complex plane and the extended function satisfies a Riemann type of functional equation then, under some other mild constraints, its nontrivial zeros have all the real part equal to 1/2.

On the other hand, dealing with cases similar to that studied by Potter and Titchmarsh, we will show that their continuous deformations represent an inexhaustible source of functions satisfying Riemann type of functional equations, not satisfying GRH and possessing double non trivial zeros on the critical line. As shown in [15], L-functions for which GRH has been formulated, do not have double zeros and therefore these continuous deformations do not represent counterexamples to GRH.

The adopted meaning of the concept of trivial zero is that which has been originally implied, namely zeros of some elementary factors of the function (which can be trivially computed). Incidentally, for the Riemann Zeta function those zeros were the real zeros of the function, fact which generated the idea that the trivial zeros should be always real. Yet, for some other L-functions, as for example those defined by non primitive Dirichlet characters, it is known that trivial imaginary zeros can exist. Also, the function (2.1) below has zeros on the critical line, namely those of the factor $(1 + \sqrt{5}/5^s)$ which can be trivially computed, hence they are trivial zeros.

As seen in [16], only adopting this approach, the trivial zeros of the derivatives of some L-functions can be unambiguously defined. On the other hand, regarding GRH, we don't need any more to make a distinction between primitive and non primitive Dirichlet characters, since the zeros associated with non primitive characters, which should be discarded, are trivial.

2 Riemann Type of Functional Equations Having Several Linearly Independent Solutions

The typical example of a pair of linearly independent L-functions $f_0(s)$ and $f_1(s)$ satisfying the same functional equation was given in [4],

where

$$f_0(s) = \left(1 + \frac{\sqrt{5}}{5^s}\right)\zeta(s) \tag{2.1}$$

which coincides for $s > 1$ with the sum of the Dirichlet series with periodic coefficients:

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1+\sqrt{5}}{5^s} + ... \tag{2.2}$$

and $f_1(s)$ is $L(5,3,s)$, obtained by analytic continuation to the whole complex plane of the Dirichlet series with periodic coefficients

$$1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s} + \frac{0}{5^s} + ... \tag{2.3}$$

It is obvious that (2.2) and (2.3) are linearly independent, since (2.2) tends to ∞ as $s- > 1$, while (2.3) is convergent at $s = 1$. The functions f_0 and f_1 satisfy both the Riemann functional equation:

$$f(s) = W(s)\overline{f(1-\overline{s})}, \tag{2.4}$$

where $W(s) = 5^{(1/2)-s}2(2\pi)^{s-1}\Gamma(1-s)\sin\frac{\pi s}{2}$.

Since both functions are real on the real axis we have $\overline{f_k(1-\overline{s})} = f_k(1-s)$, $k = 0, 1$ and if $f_k(\sigma_0 + it_0) = 0$, then necessarily $f_k(\sigma_0 - it_0) = 0$ and due to (2.4) if $W(\sigma_0 + it_0) \neq 0$ we have also $f_k(1 - \sigma_0 + it_0) = 0$. It can be easily checked that any linear combination with real coefficients $f = \alpha_0 f_0 + \alpha_1 f_1$ of f_0 and f_1 is real on the real axis and f satisfies also (2.4) with $\overline{f(1-\overline{s})} = f(1-s)$.

If one of the coefficients α_k is not real then $f(s)$ is not real for real s, which implies that $f(s)$ does not satisfy (2.4). Indeed, $\overline{f(1-\overline{s})} = \overline{\alpha_0}\overline{f_0(1-\overline{s})} + \overline{\alpha_1}\overline{f_1(1-\overline{s})}$ hence $f(s) \neq W(s)\overline{f(1-\overline{s})}$.

Let us denote

$$\varphi_\tau(s) = (1-\tau)f_0(s) + \tau f_1(s), \qquad 0 \leq \tau \leq 1, \tag{2.5}$$

where f_0 and f_1 are the functions (2.2) and (2.3).

We give to the word *deformation* used in [3] the following precise meaning: the family of functions $\{\varphi_\tau(s) \mid 0 \leq \tau \leq 1\}$ given by the formula (2.5) represents a *continuous deformation* of $f_0(s)$ into $f_1(s)$ if for any compact set $K \subset \mathbb{C}\backslash\{1\}$ and any $\tau_0 \in [0,1]$ we have $\lim\limits_{\tau\to\tau_0} \varphi_\tau(s) = \varphi_{\tau_0}(s)$ uniformly in K. Obviously $\varphi_0(s) = f_0(s)$, respectively $\varphi_1(s) = f_1(s)$. Sometimes we will call a particular function $\varphi_\tau(s)$ a deformation of $f_0(s)$ or of $f_1(s)$.

Theorem 2.1. *For every $\tau \in [0,1]$, the functions $\varphi_\tau(s)$ given by* (2.5) *satisfy* (2.4) *and define a continuous deformation of $f_0(s)$ into $f_1(s)$.*

Proof: Since $f_0(s)$ and $f_1(s)$ satisfy (2.4), and $f_k(\overline{s}) = \overline{f_k(s)}$ we have:

$$\begin{aligned}\varphi_\tau(s) &= (1-\tau)f_0(s) + \tau f_1(s) = (1-\tau)W(s)f_0(1-s) + \tau W(s)f_1(1-s)\\ &= W(s)[(1-\tau)f_0(1-s) + \tau f_1(1-s)] = W(s)\varphi_\tau(1-s),\end{aligned}$$

hence $\varphi_\tau(s)$ satisfies (2.4) for every τ, $0 \leq \tau \leq 1$.

For any compact $K \subset \mathbb{C}\backslash\{1\}$, we have that $\max_k \sup\limits_{s\in K} |f_k(s)|$ is a finite number M, due to the uniform continuity of f_k on K. Then $|\varphi_\tau(s) - \varphi_{\tau'}(s)| = |\tau' - \tau||f_0(s) - f_1(s)| \leq 2M|\tau' - \tau|$ in K. $\square$

Although there were no specific values indicated, it has been implied in [3] that some of the zeros of $\varphi_\tau(s)$ should be off critical line. However, we can prove the following:

Theorem 2.2. *For every $\tau \in [0, 1]$ the non trivial zeros of $\varphi_\tau(s)$ given by* (2.5) *have the real part equal to $\frac{1}{2}$.*

Proof: Let us notice that $f_0(s)$ has the same non trivial zeros as $\zeta(s)$ and $f_1(s)$ is the Dirichlet L-function $L(5, 3, s)$. By GRH, which we are taken as true in this paper (see [12] and [14]), the non trivial zeros of the two functions have the real part 1/2. To prove the theorem we need only to check that $\varphi_\tau(s)$ is the analytic continuation to the whole complex plane, except for the pole $s = 1$ of the corresponding linear combination of the two series, which due to the uniqueness theorem of analytic functions is obvious, and that $\lim_{\sigma->+\infty} \varphi_\tau(\sigma + it) = 1$. Indeed, this last equality is true having in view that both $f_0(s)$ and $f_1(s)$ have the limit 1 as $\sigma- > +\infty$. Replacing $f_0(s)$ by $(1 - 2^{1-s})f_0(s)$, $f_1(s)$ by $(1 - 2^{1-s})f_1(s)$ and $W(s)$ by $[(1 - 2^{1-s}) \ / \ (1 - 2^s)]W(s)$, the corresponding series of (2.1), (2.3) and (2.5) will have all the abscissa of convergence $\sigma_c = 0$. Then, by [14], Theorem 3, all the non trivial zeros of $\varphi_\tau(s)$ have the real part 1/2. $\square$

We will be dealing in the following with arbitrary L-functions (see [13]), including Dirichlet L-functions (see [12], [17], [18]). The interest in a more general setting consists also in the fact that the fascinating universality property of Riemann Zeta function extends to a wide class of functions, as shown in [19] and [20].

Theorem 2.3. *Let f_0 and f_1 be two L-functions such that $\lim\limits_{\sigma\to\infty} f_k(\sigma + it) = 1$, $k = 0, 1$ and such that they satisfy the same Riemann type of functional equation and GRH. If all the non trivial zeros of $\varphi_\tau(s)$defined by* (2.5) *are simple for every $\tau \in [0, 1]$, then for all τ, $\varphi_\tau(s)$ satisfy GRH.*

Proof: Suppose that for a certain $\tau \in [0, 1]$, $\varphi_\tau(s)$ has a zero s_τ with $Re\,(s_\tau) \neq 1/2$. By Theorem 1, $\varphi_\tau(s)$ satisfies the same functional equation as $f_k(s)$.

Then we have also $\varphi_\tau(1 - \overline{s_\tau}) = 0$. As $\tau- > 0$, s_τ and $1 - \overline{s_\tau}$ tend both to a zero s_0 of $f_0(s)$ (see next theorem), which should hence be a double zero and this is by [15] a contradiction. $\square$

Fig. 1. $Re\,(\varphi_\tau(s)) = 1/2$ **for every** τ**,** $0 \leq \tau \leq 1$

Fig. 1 illustrates this situation for the particular values of τ : 0, 0.25, 0.5, 0.75, 1 and $s \in [0, 1] \times [0, 30]$. For the seek of space economy the axes have been rotated by $\pi/2$. It can be seen how the zeros of $f_0(s)$ evolve alongside the critical line into those of $f_1(s)$ as τ varies from 0 to 1.

This phenomenon suggests a strong connection between the non trivial zeros of two L-functions satisfying the same Riemann type of functional equation, namely it indicates that the zeros of each one of them separate the zeros of the other one in the sense that the zeros of the two functions alternate or if two of them coincide, then the others alternate.

Theorem 2.4. *Let f_0 and f_1 be two L-functions such that $\lim_{\sigma\to+\infty} f_k(\sigma+it)=1$, $k=0,1$ and they satisfy the same Riemann type of functional equation. If for every τ, $0\le\tau\le 1$, $\varphi_\tau(s)$ defined by* (2.5) *satisfies GRH, then any interval $I=\{1/2+it|t_1\le t\le t_2\}$ of the critical line can be extended to an interval I' such that f_0 and f_1 have the same number of zeros in I'.*

Proof: We notice that $\varphi_\tau(s)$ defined by (2.5) satisfies also the respective functional equation. Let us follow the trajectory of one particular zero of $f_0(s)$ when τ varies from 0 to 1. We have

$$f_0(s)-\varphi_\tau(s)=\tau\left[f_0(s)-f_1(s)\right].$$

In particular, if $f_0(s_0)=0$ and $f_1(s_0)=0$, then $\varphi_\tau(s_0)=0$ for every τ, $0\le\tau\le 1$. Vice-versa, if $f_0(s_0)=0$ and $\varphi_\tau(s_0)=0$ for a value of τ, we have that $f_1(s_0)=0$ and then $\varphi_\tau(s_0)=0$ for every τ, $0\le\tau\le 1$. Suppose that for a given s_0 we have $f_0(s_0)=0$ and $f_1(s_0)\ne 0$. Then $|\varphi_\tau(s_0)|$ $=\tau|f_1(s_0)|$ and for every $\epsilon>0$ there is $\delta>0$ such that $|\varphi_\tau(s_0)|<\epsilon$ if $\tau<\delta$, which means that the disc centered at $\varphi_\tau(s_0)$ and of radius ϵ contains the origin. Thus the pre-image by $\varphi_\tau(s)$ of this disc contains at least one zero of $\varphi_\tau(s)$. The number of zeros of $\varphi_\tau(s)$ in this neighborhood must be finite, hence it make sense to look for the closest zero to s_0.

However, two different zeros, one above s_0 and the other below s_0 on the critical line might have both the smallest distance to s_0. In such a case we need to take a smaller τ and repeat the reasoning until the uniqueness condition is fulfilled. This must happen after a finite number of steps, since otherwise s_0 would be a double zero of $f_0(s)$, which by [15] is not possible. Let s_τ be the closest zero to s_0 of $\varphi_\tau(s)$ as previously defined. We can interpret $\varphi_\tau(s)$ as a deformation of $f_0(s)$ which *carried* with it the zero s_0 into a new location s_τ on the critical line. By the previous analysis, this is a *continuous motion* in the sense that any intermediate value $s_{\tau'}$ is a zero of the corresponding $\varphi_{\tau'}(s)$. At the next step, the deformation of $\varphi_\tau(s)$, which is a new deformation of $f_0(s)$ (this can be checked easily) will carry s_τ to a new location $s_{\tau''}$ and so on. At every step s_0 is moved (obviously in the same direction) into an open neighborhood on I and since I is a compact set, after a finite number of steps $\tau=1$, hence $\varphi_\tau(s)=f_1(s)$ and s_0 is carried to a zero s_1 of $f_1(s)$. A continuous deformation of $f_1(s)$ into $f_0(s)$ can be performed analogously and this process establishes a one-to-one correspondence between the zeros of $f_0(s)$ and those of $f_1(s)$ such that if an interval I for t is given we can extend it if necessary to a bigger one which contains only pairs of corresponding zeros of the two functions and the theorem is completely proved. □

Theorem 2.5. *Let $S_k^{(0)}$ and $S_k^{(1)}$ be the S_k-strips of two functions $f_0(s)$ and $f_1(s)$ from Theorem 2.4. Then every intersection $S_k^{(0)}\cap S_{k'}^{(1)}$ contains the same number of non trivial zeros of $f_0(s)$ and of $f_1(s)$ or these numbers are different by one unit.*

Proof: Let $s_k^{(0)}$ and $s_k^{(1)}$ be the intersection of the critical line with the curves Γ_k' corresponding to $f_0(s)$ respectively $f_1(s)$. Suppose $s_k^{(0)}<s_k^{(1)}$. Since $f_0(s_k^{(0)})>1$, $s_k^{(0)}$ cannot be a zero of $f_0(s)$. However, it can be a zero of $f_1(s)$. If $f_1(s_k^{(0)})=0$, then the next zero belongs to $f_0(s)$, or it is a common zero of both functions and the zeros alternate in the sense previously described. When counting the respective zeros, the numbers should be either the same, or different by a unit. Therefore the one-to-one correspondence between the zeros of the two functions can be confined to the respective intersections if we assign the extra zero when it appears to one of the adjacent intersections.□

It appears that this property extends to Davenport and Heilbronn type of functions. We illustrate this affirmation by Fig. 2 below where we took the Davenport and Heilbronn type of functions

$$f_0(s)=\frac{1}{2}[(1+0.3088766085i)L(17,2,s)+(1-0.3088766085i)L(17,16,s)]$$

and

$$f_1(s) = \frac{1}{2}[(1 - 3.237538785i)L(17, 2, s) + (1 + 3.237538785i)L(17, 16, s)].$$

For the seek of space economy, the axes are rotated by $\pi/2$. Four intersections $S_k^{(0)} \cap S_k^{(1)}$ are visible containing 6 and 5, 0 and 0, 3 and 4, respectively 6 and 5 zeros of the two functions.

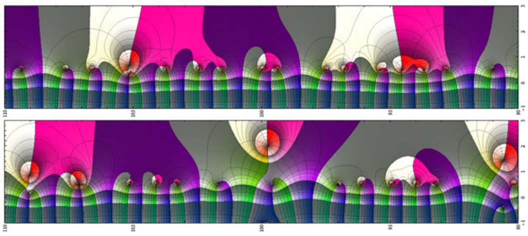

Fig. 2. $S_k^{(0)} \cap S_{k'}^{(1)}$ **contain roughly the same number of zeros**

Remark 2.1. If we denote by $N^{(0)}(T)$ and $N^{(1)}(T)$ the number of non trivial zeros of $f_0(s)$ respectively $f_1(s)$ in the interval $[0, T]$ or in the interval $[-T, 0]$ of the critical axis, then by Theorem 2.5 we have $N^{(0)}(s_k^{(0)}) = N^{(1)}(s_k^{(0)}) + \kappa$ and $N^{(0)}(s_k^{(1)}) = N^{(1)}(s_k^{(1)}) + \kappa$ for every $k \in \mathbb{Z}$, $k \neq 0$, where κ is 0 or ± 1.

Remark 2.2. An interesting question arises about some of the non trivial zeros of the function $f_1(s)$ obtained by analytic continuation of the Dirichlet series (2.3). It can be noticed that the zeros of $1 + 5^{(1/2)-s}$ are preserved during the continuous deformation of $f_0(s)$ into $f_1(s)$, in other words some of the apparently non trivial zeros of $f_1(s)$ are the same as some trivial zeros of $f_0(s)$. This apparent contradiction can be settled by noticing that $f_1(s)$ can be also factorized by $1 + 5^{(1/2)-s}$. Indeed

$$1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s} + \frac{0}{5^s} + \frac{1}{6^s} - \frac{1}{7^s} - \frac{1}{8^s} + \frac{1}{9^s} + \frac{0}{10^s} + ... =$$

$$\left(1 + \frac{\sqrt{5}}{5^s}\right)\left(1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s} - \frac{\sqrt{5}}{5^s} + \frac{1}{6^s} - \frac{1}{7^s} - \frac{1}{8^s} + \frac{\sqrt{5}}{10^s} + ...\right).$$

This last series has the same abscissa of convergence as the series (2.3) and it can be continued analytically to the whole complex plane to an L-function $f_2(s)$. Hence the true non trivial zeros of $f_1(s)$ are only those which are non trivial also for $f_2(s)$.

The Figs. 1 and 2 give us a lot of visual information which is waiting to get a rigorous proof. On the other hand, despite of all the expectations, under the assumptions adopted in [4], the affirmation in [4] and [18] that given two linearly independent solutions $f_1(s)$ and $f_2(s)$ of a Riemann-type of functional equation, the function $f(s) = f_1(s_0)f_2(s) - f_2(s_0)f_1(s)$ satisfies the respective equation must be false. Obviously $f(s)$ has the arbitrary zero s_0,which can be taken off critical line. We dealt with this topic in the opinion paper[11].

3 More Dirichlet L-Functions Satisfying the Same Functional Equation

Let $L(s;\chi_k)$ be the Dirichlet L-function defined by a Dirichlet character χ_k modulo q, i.e.

$$L(s;\chi_k) = \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^s} \tag{3.1}$$

where $\chi_k(n)$ is the k-th Dirichlet character modulo q. Then $L(s;\chi_k)$ satisfies (see [9]) the functional equation

$$L(s;\chi) = \epsilon(\chi)W(s)L(1-s;\overline{\chi}), \tag{3.2}$$

where $W(s) = 2^s q^{(1/2)-s}\pi^{s-1}\Gamma(1-s)\sin\frac{\pi}{2}(s+\kappa)$ and $\kappa = 0$ if $\chi(-1) = 1$ (χ is even), $\kappa = 1$ if

$\chi(-1) = -1$ (χ is odd), $\epsilon(\chi) = \tau(\chi)/i^{\kappa}\sqrt{q}$ and $\tau(\chi) = \Sigma_{k=1}^{q}\chi(k)Exp\{2\pi i/q\}$, $|\epsilon(\chi)| = 1$

Obviously, if χ_k and $\chi_{k'}$ have different parities, the corresponding functional equations are different and if χ_k and $\chi_{k'}$ have the same parity in order for these functional equations to coincide, we must have $\tau(\chi_k) = \tau(\chi_{k'})$. In other words, $L(s;\chi_k)$ and $L(s;\chi_{k'})$ satisfy the same functional equation if and only if $\tau(\chi_k) = \tau(\chi_{k'})$. Yet, the line matrices having the components $\chi_k(n)Exp\{2n\pi i/q\}$ are linearly independent and therefore $k \neq k'$ implies $\tau(\chi_k) \neq \tau(\chi_{k'})$. Therefore, no two Dirichlet L-functions can satisfy the same Riemann type of functional equation. In order to find L-functions satisfying the same functional equation we need to expand on the example from the section 2, considering an arbitrary modulus q. So, let

$$f_0(s) = [1 + q^{(1/2)-s}]\zeta(s) \tag{3.3}$$

and let us notice that $f_0(s)$ satisfies the functional equation

$$f(s) = W(s)f(1-s), \text{ where } W(s) = q^{s-1/2}2(2\pi)^{s-1}\Gamma(1-s)\sin\frac{\pi s}{2} \tag{3.4}$$

Then we should look for even real primitive non principal characters $\chi(\mod q)$, which generate Dirichlet L-functions satisfying the functional equation (3.4). For $q = 5$, such a candidate is $\chi_3(\mod 5)$, which we have already encountered. Then, ordered by increasing q the next characters of interest are: $\chi_2(\mod 8)$, $\chi_3(\mod 10)$, $\chi_4(\mod 12)$, $\chi_7(\mod 13)$, $\chi_3(\mod 15)$, $\chi_9(\mod 17)$, $\chi_{10}(\mod 21)$, etc. We cannot find a formula generating all these characters, but it is reasonable to assume that their sequence is infinite.

There are also odd quadratic characters satisfying a similar equation, namely $\chi_2(\mod 3)$, $\chi_2(\mod 4)$, $\chi_2(\mod 6)$, $\chi_4(\mod 7)$, $\chi_6(\mod 11)$, $\chi_3(\mod 12)$, $\chi_4(\mod 14)$, $\chi_5(\mod 15)$, $\chi_{10}(\mod 19)$, etc. We checked that for all these characters, even and odd, $\epsilon(\chi) = 1$, but since $\kappa = 1$ for odd characters, we have $\cos\frac{\pi s}{2}$ instead of $\sin\frac{\pi s}{2}$ in the expression of $W(s)$.

The Dirichlet L-functions $L(q,k,s)$ defined by quadratic even characters χ_k satisfy each one the equation (3.4) corresponding to the respective q. The same equation is satisfied by $(1+q^{(1/2)-s})\zeta(s)$. To find a similar property for the odd characters, we need to replace the right hand side in the formula (3.3) by a Davenport and Heilbronn function modulo q (see [11]), which contains in (3.4) the right trigonometric function, i.e. which is defined using odd complex conjugate Dirichlet characters modulo q in the formula:

$$f_0(s) = \frac{1}{2}\{[L(s;\chi) + L(s;\overline{\chi})] + i\tan\theta[L(s;\chi) - L(s;\overline{\chi})]\} \tag{3.5}$$

and where $\epsilon(\chi) = e^{2i\theta}$.

On the other hand, if χ and $\overline{\chi}$ are even complex conjugate Dirichlet characters and for the respective q there is an even quadratic Dirichlet character χ_k, then $L(q,k,s)$ satisfies also the functional equation (3.4).

Theorem 3.1. *For q and k from the previous sequence the function $L(q,k,s)$ has some trivial zeros on the critical line, namely the zeros of $1+q^{(1/2)-s}$ and its non trivial zeros are obtained by moving to their location some zeros of $f_0(s)$ from (3.3), respectively (3.5) by a continuous deformation of $f_0(s)$ into $L(q,k,s)$. The trivial zeros on the critical line of the two functions are the same and they are preserved through deformation. Any interval I for t can be extended to I' such that $f_0(s)$ and $L(q,k,s)$ have the same number of zeros in I'.*

The proof of this theorem is similar to that of Theorem 2.3 and we skip it. It is important however to notice that since the non trivial zeros of $f_0(s)$ from (3.3) and those of $L(q,k,s)$ are on the critical line, the trajectories of these zeros must remain on the critical line, as seen in Theorem 2.3. Not the same thing happens with those non trivial zeros of f_0 from (3.5) which are off critical line. Yet, the deformation $\varphi_\tau(s)$ of $f_0(s)$ continues to satisfy a Riemann type of functional equation, which implies that the non trivial zeros off critical line must be symmetric two by two with respect to the critical line. In other words, every couple of zeros situated in horizontal position is moved continuously when τ varies from 0 to 1 into a couple of zeros situated in vertical position. Such a motion is possible only if for a value of τ the two zeros coincide, i.e. the respective $\varphi_\tau(s)$ has a double zero. Therefore the continuous deformations of the Davenport and Heilbronn type of functions represent an unlimited source of functions which are obtained by analytic continuation to the whole complex plane of Dirichlet series, satisfying Riemann type of functional equations and possessing double zeros. The Davenport and Heilbronn function has been obtained by using odd Dirichlet characters modulo 5, in which case no quadratic odd Dirichlet character exists, hence the idea of double zeros could not appear.

However, this could have been done if somebody realized that the equation $\epsilon(\chi) = e^{2i\theta}$ has two solutions, namely $\tan\theta = 0.2840790438...$ and $\tan\theta = -3.520147022...$ Only the first one is mentioned in all the publications on this topic (see [1], [2], [3], [4], [17]). Using both of them, two linearly independent Davenport and Heilbronn type of functions could have been defined satisfying both the same Riemann-type of functional equation and a continuous deformation of one into the other could have provided an example of function having double non trivial zeros. We came with such an example for the Davenport and Heilbronn type of functions defined by $L(17,2,s)$ and $L(17,16,s)$ in Fig. 2. For these functions $\tan\theta$ is $-0.3088766085...$ and $3.237538785...$

We give next a short list of Dirichlet characters which generate Davenport and Heilbronn type of functions having deformations which display double zeros. It is obvious that the long list is infinite. On the other hand, for every modulus q big enough there are several couples of complex conjugate Dirichlet characters having the same parity as the quadratic non principal character modulo q, therefore several Dirichlet L-functions to be used for the purpose of producing Davenport and Heilbronn type of functions which have continuous deformations into Dirichlet L-functions. We chose the first three prime moduli greater than 5 and computed for them the corresponding values of $\tan\theta$ to offer a ready-to-use sample of L-functions possessing double zeros.

For $q = 7$, we have the odd complex conjugate Dirichlet characters χ_2 and χ_6 and the quadratic odd Dirichlet character χ_4.We have found $\tan\theta = 0.66518189...$ or $\tan\theta = -1.503348205...$ For the first value we present in Fig. 3 below an illustration of the phenomenon described above.

To obtain Fig. 3 we needed to locate first approximately the symmetric zeros with respect to the critical line of the function (3.5) where $q = 7$ and $\chi = \chi_2 (\mod 7)$. Taking the pre-image of the real axis by $f_0(s)$ for $t \in [30,60]$, three pairs of symmetric zeros have been detected, namely when t is approximately 31.5, 49.5 and 59, as it can be seen in Fig. 4 below. We chose $t = 31.5$. The first row in Fig. 3 represents instances of the continuous deformation of $f_0(s)$ into $L(7, 4, s)$ corresponding to the values 0, 0.25, 0.5, 0.75, 1 of the parameter τ. We notice that the switch from horizontal setting to vertical setting of the respective symmetric zeros happens for a τ in the interval $[0.25, 0.5]$. The second row portrays the instances of that continuous deformation corresponding to five values of τ in this last interval. It looks like $\varphi_{0.34375}$ has a double zero at $0.5 + 31.5i$.

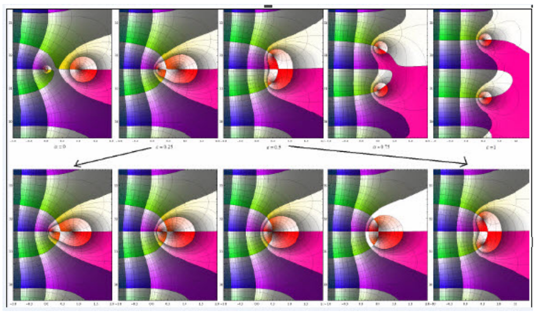

Fig. 3. Double zero obtained by trial and error for $\tau = 0.34375$

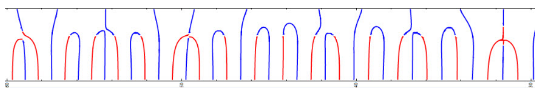

Fig. 4. The pre-image of the real axis by $f_0(s)$ **in the box** $[-2, 2] \times [30, 60]$

When the distance between the symmetric zeros with respect to the critical line is small, as in the case of $t = 49.5$, the change of configuration in the neighborhood of the double zero can be sudden, which makes difficult the estimation of the value of τ corresponding to a double zero. As it can be seen in Fig. 5, a bound of 10^{-8} for the error of τ was still not good enough for an accurate location of that double zero. Yet, there is no doubt that this zero exists.

$\tau = 0.046875$ $\tau = 0.046875002$

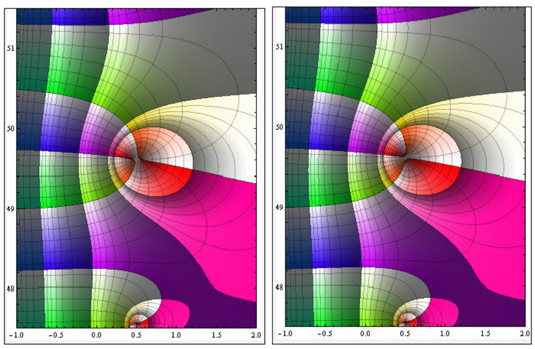

Fig. 5. Sudden change of configuration in a neighborhood of a double zero.

The following table provides ready-to-use data for computer experimentation in hunting for double zeros of continuous deformations of Dirichlet L-functions.

Table. 1. Data for computer experimentation

q	complex conjugate characters	quadratic character	$\tan\theta_1$	$\tan\theta_2$
7	χ_2 and χ_6	χ_4(odd)	0.6651818899...	$-1.503348205...$
7	χ_3 and χ_5	χ_4	$-0.2342612812...$	4.268737860...
11	χ_2 and χ_{10}	χ_6 (odd)	$-0.7434740182...$	1.345036915...
11	χ_4 and χ_8	χ_6 (odd)	0.3638119069...	$-2.748673095...$
11	χ_3 and χ_9	χ_6	0.3381131754...	$-2.957589566...$
11	χ_5 and χ_7	χ_6	0.4810328598...	$-2.078860060...$
13	χ_3 and χ_{11}	χ_7 (even)	0.2748333011...	$-3.642777128...$
13	χ_5 and χ_9	χ_7 (even)	0.7724331007...	$-1.294610496...$
13	χ_2 and χ_{12}	χ_7	0.5602830041...	$-1.784812305...$
13	χ_4 and χ_{10}	χ_7	0.7420558960...	$-1.347607245...$
13	χ_6 and χ_8	χ_7	0.9621130197...	$-1.039378929...$

In every case the Davenport and Heilbronn type of function has the form

$$f_0(s) = \frac{1}{2}[(1 - i\tan\theta)L(q, k, s) + (1 + i\tan\theta)L(q, q + 1 - k, s)].$$

For $q = 7, 11$ and 13 and for odd Dirichlet characters a continuous deformation of the corresponding $f_0(s)$ into $L(7, 4, s)$, $L(11, 6, s)$ and respectively $L(13, 7, s)$ can be performed. Similarly, for even Dirichlet characters, a continuous deformation of the corresponding $f_0(s)$ into $[1 + 7^{(1/2)-s}]\zeta(s)$, $[1 + 11^{(1/2)-s}]\zeta(s)$ and respectively $[1 + 13^{(1/2)-s}]\zeta(s)$ can be performed. Each one of these deformations will display a double zero for some τ. Continuous deformations can be considered between two functions such that none of them satisfies GRH. Then we have:

Theorem 3.2. *Let $f_0(s)$ and $f_1(s)$ satisfy the same Riemann-type of functional equation, but do not satisfy GRH. A continuous deformation $\varphi_\tau(s)$ of $f_0(s)$ into $f_1(s)$ can carry two zeros of $f_0(s)$ symmetric with respect to the critical line into two zeros of $f_1(s)$ on the critical line and vice-versa. If this is the case, then there is τ_0, $0 < \tau_0 < 1$ such that $\varphi_{\tau_0}(s)$ has a double zero, which is located on the critical line. There is a one-to-one correspondence between the zeros of $f_0(s)$ and $f_1(s)$ such that every intersection of fundamental domains of the two functions contains roughly the same number of zeros of each one of them.*

Proof: A double zero of $\varphi_{\tau_0}(s)$, if it exists, is the limit as $\tau -> \tau_0$ of symmetric zeros with respect to the critical line of $\varphi_\tau(s)$, thus it must be located on the critical line. The one-to-one correspondence between the zeros of $f_0(s)$ and $f_1(s)$ is a corollary of Theorem 5. In particular, the number of zeros of $f_0(s)$ and of $f_1(s)$ in the intersection of two fundamental domains of each one of the two functions can be different by at most one unit.

4 Conclusions

A closer look into a method intended to provide counterexamples to GRH allowed us to reveal properties common to a whole class of functions labelled as Davenport and Heilbronn type of functions.

Continuous deformations of these functions represent an inexhaustible source of functions satisfying Riemann type of functional equations and having off critical line non trivial zeros. They also exhibit double non trivial zeros, the hunt for which was unsuccessful until now.

However, they are not counterexample for GRH, since they do not belong to the class of functions for which GRH has been formulated.

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Competing Interests

Authors have declared that no competing interests exist.

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