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On divisible and pure multigroups and their properties

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Abstract: The theory of multigroups is a generalized group's theoretic notions in multiset framework. Although myriad of researches have been done in multigroup theory, but some group's analogue concepts have not been investigated in multigroup setting. In this paper we propose the notions of divisible and pure multigroups and characterize some of their properties. It is established that the image and preimage of homomorphism of divisible and pure multigroups are divisible and pure multigroups. The nexus between divisible and pure multigroups and that of divisible and pure groups are instituted using the concept of cuts of multigroups.

1. Introduction

The term multisets as buttressed by Knuth [1], was first recommended by N. G. de Bruijn (cf. [2]) in a communication to D. E. Knuth, as a generalization of classical sets, by violating the principle of distinct collection of elements in set theory. Both the theory and applications of multiset have been extensively studied in [3–8]. In a way to apply group theory to multisets, Nazmul *et al.,* [9] proposed the concept of multigroups in multisets framework and presented a number of results, just as group theory stem out of set theory. A comprehensive account on the concept of multigroups has been carried out, and it has been established that multigroup via multiset is a generalization of group theory [10].

Several researches have emerged on the concept of multigroup theory. Some results on multigroups which cut across some homomorphic properties were explored in [11,12]. The notions of upper and lower cuts of multigroups were introduced and discussed in details with some number of results in [13], and the notions were extended to homomorphism in [14]. Some group's analogous concepts like normal subgroups, characteristic subgroups, direct product, cosets, factor groups and group actions, etc. have been established in multigroup context [15–27]. The idea of commutators in multigroup context has been studied in [28]. The notion of ring structure from multiset context has been proposed and some related results deduced [29].

Though numerous constructs in group theory have been established in multigroup structure, some constructs have not been considered in multigroup context. The notions of divisible and pure groups have been hitherto studied in other non-classical groups like fuzzy groups and intuitionistic fuzzy groups, respectively [30–32]. This paper attempts to strengthen multigroup theory by characterizing the constructs of divisible and pure multigroups which are the applications of divisible and pure groups in multisets. We first defined a special multisets of a group to boost the introduction of divisible and pure multigroups. We study the homomorphic image and preimage of divisible and pure multigroups and found that they are divisible and pure multigroups. The correlation between divisible and pure multigroups and that of divisible and pure groups are established using the idea of multigroup cuts. The rest of the paper is outlined as follows; Section 2 presents the ideas of multisets, multigroups and some helpful existing results. Section 3 discusses and characterizes divisible multigroups. Finally, Section 4 gives a precise conclusion with area of future research.

2. Preliminaries

Throughout this paper X denotes non-empty set and G denotes an additive group with identity element 0.

Definition 1. [8] Let X be a set. A multiset A over X is just a pair $\langle X, C_A \rangle$, where

$$C_A : X \to \mathcal{N} = \{0, 1, 2, ...\}$$

is a function, such that for $x \in X$ implies $A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$ denoted the number of times an object x occur in A. Whenever $C_A(x) = 0$, implies $x \notin X$. The set X is called the ground or generic set of the class of all multisets containing objects from X.

Definition 2. [7] Let X be the set from which multisets are constructed. The multiset X^n is the set of all multisets of X such that no element occurs more than n times. Likewise, the multiset X^∞ is the set of all multisets of X such that there is no limit on the number of occurrences of an element.

Definition 3. [7] Let A, B be multisets of X. Then

  (i) $A = B \iff C_A(x) = C_B(x) \; \forall x \in X$,
 (ii) $A \subseteq B \iff C_A(x) \leq C_B(x) \; \forall x \in X$,
(iii) $A \cap B \implies C_{A \cap B}(x) = C_A(x) \wedge C_B(x) \; \forall x \in X$,
(iv) $A \cup B \implies C_{A \cup B}(x) = C_A(x) \vee C_B(x) \; \forall x \in X$,
 (v) $A \oplus B \implies C_{A \oplus B}(x) = C_A(x) \oplus C_B(x) \; \forall x \in X$,

where $\wedge$ and $\vee$ denote minimum and maximum respectively.

Definition 4. [9] Suppose $\{A_i\}_{i \in I}$ is an arbitrary family of multisets of X. Then $\bigcap_{i \in I} A_i = \bigwedge_{i \in I} C_{A_i}(x) \; \forall x \in X$ and $\bigcup_{i \in I} A_i = \bigvee_{i \in I} C_{A_i}(x) \; \forall x \in X$.

Definition 5. [9] A multiset A of G is called a multigroup if

  (i) $C_A(x + y) \geq C_A(x) \wedge C_A(y) \; \forall x, y \in G$,
 (ii) $C_A(-x) = C_A(x) \; \forall x \in G$.

From (i) and (ii), we have $C_A(x - y) \geq C_A(x) \wedge C_A(y) \; \forall x, y \in G$. In fact, in a multigroup A of G, $C_A(0) \geq C_A(x) \; \forall x \in G$.

Definition 6. [10] A multigroup A of G is said to be constant or has a constant count function if $C_A(x) = C_A(y)$ $\forall x, y \in G$.

Definition 7. [10] A multigroup A of G is said to be commutative if and only if

$$C_A(x + y) = C_A(y + x) \; \forall x, y \in G.$$

Proposition 1. *[9] If A and B are multigroups of G, then (i) $A \cap B$ and $A \oplus B$ are multigroups of G, (ii) $A \cup B$ is a multigroup of G provided $A \subseteq B$.*

Theorem 1. *[13] Let A be a multigroup of G. Then, the sets A_n and A^n defined by*

$$A_n = \{x \in G \mid C_A(x) \geq n\}$$

and

$$A^n = \{x \in G \mid C_A(x) \leq n\}$$

where $n \in \mathbb{N}$, are subgroups of G for $n \leq C_A(0)$ and $n \geq C_A(0)$, respectively.

Theorem 2. *[22] Let A be a multiset of G and $n \in \mathbb{N}$.*

  (i) *If every A_n for $n \leq C_A(0)$ is a subgroup of G, then A is a multigroup of G.*
 (ii) *If every A^n for $n \geq C_A(0)$ is a subgroup of G, then A is a multigroup of G.*

Definition 8. [12] Let $f : G \to G'$ be a homomorphism of groups. Suppose A and B are multigroups of G and G', respectively. Then, f induces a homomorphism from A to B which satisfies

  (i) $C_A(f^{-1}(y_1 y_2)) \geq C_A(f^{-1}(y_1)) \wedge C_A(f^{-1}(y_2)) \; \forall y_1, y_2 \in G'$,

(ii) $C_B(f(x_1x_2)) \geq C_B(f(x_1)) \wedge C_B(f(x_2)) \ \forall x_1, x_2 \in G,$

where

(i) the image of A under f, denoted by $f(A)$, is a multiset of G' defined by

$$C_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} C_A(x), & f^{-1}(y) \neq \varnothing \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in G'$.

(ii) the preimage of B under f, denoted by $f^{-1}(B)$, is a multiset of G defined by

$$C_{f^{-1}(B)}(x) = C_B(f(x)) \ \forall x \in G.$$

Proposition 2. *[12] Let $f: G \to G'$ be a homomorphism of groups, A and B be multigroups of G and G', respectively. Then $f(A)$ is multigroup of G' and if f is an isomorphism, then $f^{-1}(B)$ is a multigroup of G.*

3. Divisible multigroups and pure multigroups

This section introduces the concepts of divisible and pure multigroups and presents some of their properties.

3.1. Some properties of divisible multigroups

Recall that an abelian group G is divisible if for every positive integer n and any $x \in G$, $\exists \, y \in G$ such that $ny = x$. Equivalently, an abelian group G is divisible if and only if $nG = G$, where n is any positive integer and $nG = \{nx | x \in G\}$.

Definition 9. Let A be a multiset of G. Then, the set nA, where n is a positive integer is defined as

$$C_{nA}(x) = \begin{cases} \bigvee_{x=ny} C_A(y), & x \in nG \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 3. *Suppose A and B are multisets of G, then*

(i) $A \subseteq B \implies nA \subseteq nB,$
(ii) $n(A \cap B) = nA \cap nB,$
(iii) $n(A \cup B) = nA \cup nB,$
(iv) $n(A \oplus B) = nA \oplus nB.$

Proof. The proof of (i) is trivial. Now, we prove (ii) as follows. Suppose $x \notin nG$, then $C_{n(A \cap B)}(x) = 0 = C_{nA \cap nB}(x)$. Again, assume $x \in nG$, then we have

$$\begin{aligned} C_{nA \cap nB}(x) &= C_{nA}(x) \wedge C_{nB}(x) \\ &= \bigvee_{x=ny} C_A(y) \wedge \bigvee_{x=ny} C_B(y) \\ &= \bigvee_{x=ny} [C_A(y) \wedge C_B(y)] \\ &= \bigvee_{x=ny} C_{A \cap B}(y) \\ &= C_{n(A \cap B)}(x). \end{aligned}$$

Hence, $n(A \cap B) = nA \cap nB$. The proofs of (iii) and (iv) follow directly from Definitions 3, 9 and (ii). $\square$

Theorem 3. *Suppose $f: G \to G'$ is a homomorphism such that A is multiset of G. Then for any $n \in \mathbb{N}$, $f(nA) = nf(A)$.*

Proof. Suppose $x \notin nG$, then $C_{f(nA)}(x) = 0 = C_{nf(A)}(x)$. Again, suppose we have $x \in nG$. If $z \in G'$ and $z = f(x)$, then

$$
\begin{aligned}
C_{f(nA)}(z) &= \bigvee_{z=f(x)} C_{nA}(x) = \bigvee_{z=f(x)} \bigvee_{x=nw} C_A(w) \\
&= \bigvee_{z=f(x),\, x=nw} C_A(w) = \bigvee_{z=f(nw)} C_A(w) \\
&= \bigvee_{z=nf(w)} C_A(w) = \bigvee_{z=ny} \bigvee_{y=f(w)} C_A(w) \\
&= \bigvee_{z=f(nw)} C_{f(A)}(y) = C_{nf(A)}(z).
\end{aligned}
$$

Hence, $f(nA) = nf(A)$. $\square$

Theorem 4. *Let f be a homomorphism from G to G' such that B is a multiset of G'. Then for any $n \in \mathbb{N}$, $nf^{-1}(B) \subseteq f^{-1}(nB)$. Moreover if f is an isomorphism, $nf^{-1}(B) = f^{-1}(nB)$.*

Proof. Let $x \in G$. If $x \notin nG$, then $C_{nf^{-1}(B)}(x) = 0 \leq C_{f^{-1}(nB)}(x)$. If $x \in nG$, then $f(x) \in nG'$. Thus

$$
\begin{aligned}
C_{nf^{-1}(B)}(x) &= \bigvee_{x=nw} C_{f^{-1}(B)}(w) = \bigvee_{x=nw} C_B(f(w)) \\
&\leq \bigvee_{f(x)=f(nw)} C_B(f(w)) \leq \bigvee_{f(x)=ny} C_B(y) \\
&= C_{nB}(ny) = C_{nB}(f(x)) \\
&= C_{f^{-1}B}(x).
\end{aligned}
$$

Hence, $nf^{-1}(B) \subseteq f^{-1}(nB)$. Suppose f is an isomorphism, then by using the argument in Theorem 3, we have

$$
\begin{aligned}
C_{f^{-1}(nB)}(x) &= C_{f^{-1}(n(f(f^{-1}(B))))}(x) = C_{f^{-1}(f(n(f^{-1}(B))))}(x) \\
&\leq C_{n(f^{-1}(B))}(x),
\end{aligned}
$$

so $f^{-1}(nB) \subseteq nf^{-1}(B)$. Therefore, $nf^{-1}(B) = f^{-1}(nB)$. $\square$

Definition 10. Let G be an abelian group. A multigroup A of G is called divisible if $C_{nA}(x) = C_A(x) \ \forall x \in G$ for every positive integer n. Equivalently, if $nA = A$ for every positive integer n.

Example 1. A multigroup of (i) an additive group of rational numbers $\mathbb{Q}$ is a divisible multigroup, (ii) a group of complex roots of unity of degrees p^k, $k = 1, ..., n$, where p is a prime number is a divisible multigroup.

Remark 1. (i) Every multigroup of a divisible group is a divisible commutative multigroup. (ii) If $G = \mathbb{Q}$, then every divisible multigroup of G has a constant count function over $\mathbb{Q} - \{0\}$.

Proposition 4. *Suppose $f : G \to G'$ is a homomorphism of groups, and A be a divisible multigroup of G. Then, the homomorphic image of A is a divisible multigroup of G'.*

Proof. From Theorem 2, we see that $f(A)$ is a multigroup of G'. Thus, we have $nf(A) = f(nA) = f(A)$, for every $n \in \mathbb{N}$ (Theorem 3). Hence, $f(A)$ is a divisible multigroup of G. $\square$

Proposition 5. *Let $f : G \to G'$ be an isomorphism of groups, and B be a divisible multigroup of G'. Then, the homomorphic preimage of B is a divisible multigroup of G.*

Proof. By synthesizing Theorems 2 and 4, it follows that $f^{-1}(B)$ is a divisible multigroup of G. $\square$

Definition 11. A multigroup $\mathtt{A}$ of G is called p-divisible if and only if $C_{p^k A}(x) = C_A(x) \ \forall x \in G$ where $k \in \mathbb{N}$ and p is a prime.

Proposition 6. *Let A be a multigroup of G. Then A is divisible if and only if it is p-divisible for every prime p.*

Proof. Suppose that A is divisible multigroup of G. Certainly, it is p-divisible since $p^k \in \mathbb{N}$.

Conversely, assume A is a p-divisible multigroup of G for every prime p. Then for every $n \in \mathbb{N}$, we get $n = p_1 p_2 ... p_m$ where p_i (for $i = 1, ..., m$) is prime. Thus, $nA = (p_1 p_2 ... p_m)A = \beta A = A$ since $n = \beta$. So, A is a divisible multigroup of G. $\square$

Theorem 5. *Let A be a divisible multigroup of G. Then, the following are divisible subgroups of G:*

(i) A_n, $n \in \mathbb{N}$ *for* $n \leq C_A(0)$.
(ii) A^n, $n \in \mathbb{N}$ *for* $n \geq C_A(0)$.

Proof. (i) By Theorem 1, A_n is a subgroup of G. Let $x \in A_n$ and $n \in \mathbb{N}$. Since $\bigvee_{ny=x} C_A(y) = C_A(x) \geq n$, then it follows that $y \in A_n$ with $ny = x$. Hence, A_n is a divisible subgroup of G.

(ii) Similarly, A^n is a subgroup of G by Theorem 1. If $x \in A^n$ and $n \in \mathbb{N}$. Then, since $\bigvee_{ny=x} C_A(y) = C_A(x) \leq n$, it follows that $y \in A^n$ such that $ny = x$, and the result follows.
$\square$

Theorem 6. *Let A be a multiset of G bounded by G^n such that $C_A(0) = n$ and every A_n, $n \in \mathbb{N}$, is a divisible subgroup of G. Then A is a divisible multigroup of G.*

Proof. From the given hypotheses, it follows that A is a multigroup of G by Theorem 2. Let $x \in G$ and $C_A(x) = n$. Since A_n is a divisible subgroup of G for every $n \in \mathbb{N}$, $\exists y \in A_n$ such that $ny = x$, hence $C_A(y) \geq C_A(x) = n$. But $C_A(y) \leq C_A(x)$ since A is a multigroup of G. Hence, $\bigvee_{ny=x} C_A(y) = C_{nA}(x)$, for $x \in nG$ and so, $nA = A$. $\square$

Corollary 7. *If A is a multiset of G bounded by G^n such that $C_A(0) = n$ and every A^n, $n \in \mathbb{N}$, is a divisible subgroup of G. Then A is a divisible multigroup of G.*

Proof. Combining Theorems 2 and 6, the result holds. $\square$

Theorem 8. *Suppose $\{A_i\}_{i \in I}$ is a family of divisible multigroups of G. Then $\bigcap_{i \in I} A_i$ is a divisible multigroup of G.*

Proof. By Proposition 1, $\bigcap_{i \in I} A_i$ is a multigroup of G. Assume that every A_i is divisible, then for $x \in nG$ we get

$$
\begin{aligned}
C_{n(\bigcap_{i \in I} A_i)}(x) &= \bigvee_{x=ny} \bigwedge_{i \in I} C_{A_i}(y) = \bigwedge_{i \in I} \bigvee_{x=ny} C_{A_i}(y) \\
&= \bigwedge_{i \in I} C_{nA_i}(x) = C_{\bigcap_{i \in I}(nA_i)}(x) \\
&= C_{\bigcap_{i \in I} A_i}(x).
\end{aligned}
$$

If $x \notin nG$, then $C_{\bigcap_{i \in I} A_i}(x) = 0 = C_{n(\bigcap_{i \in I} A_i)}(x)$. Hence, $\bigcap_{i \in I} A_i$ is divisible. $\square$

Theorem 9. *Suppose $\{A_i\}_{i \in I}$ is a family of divisible multigroups of G. Then*

(i) $\bigcup_{i \in I} A_i$ *is a divisible multigroup of G if* $A_1 \subseteq A_2 \subseteq ... \subseteq A_n$, $i = 1, ..., n$.
(ii) $\Sigma_{i \in I} A_i$ *is a divisible multigroup of G.*

Proof. By using Proposition 1 and following the same logic in Theorem 8, the proofs of (i) and (ii) are established. $\square$

Theorem 10. *Suppose A is a multigroup of a divisible cyclic group G, then A is constant.*

Proof. Let $G =< a >$ and $x \in G$. Then $x = pa$, $p \in \mathbb{N}$. Suppose that $C_A(pa) > C_A((p+1)a)$. Because G is divisible $\exists\, y \in G$ such that $py = a$. But also $\exists\, q \in \mathbb{N}$ such that $y = qa$, and so

$$
\begin{aligned}
C_A(pa) \quad &> \quad C_A((p+1)a) = C_A((p+1)py) \\
&= \quad C_A((p+1)pqa) = C_A((p+1)q(pa)) \\
&\geq \quad C_A(pa) \wedge C_A(pa) \wedge ... \wedge C_A(pa) \\
&= \quad C_A(pa),
\end{aligned}
$$

which is a contradiction.

   Also, if $C_A((p+1)a) > C_A(pa)$. Then $\exists\, z \in G$ such that $(p+1)z = a$, $z = ra$, $r \in \mathbb{N}$ since G is divisible. Thus

$$
\begin{aligned}
C_A((p+1)a) \quad &> \quad C_A(pa) = C_A(p(p+1)ra) = C_A(pr(p+1)a) \\
&\geq \quad C_A((p+1)a) \wedge C_A((p+1)a) \wedge ... \wedge C_A((p+1)a) \\
&= \quad C_A((p+1)a),
\end{aligned}
$$

which is also a contradiction. Hence, $C_A((p+1)a) = C_A(pa)$. Since x is arbitrary, if $x = a$, then

$$
C_A(a) = C_A(2a) = C_A(3a) = ...,
$$

and so A is a constant multigroup of a divisible cyclic group G. $\square$

3.2. Some properties of pure multigroups

   Recall that a subgroup H of a group G is called pure if $nH = H \cap nG$, for every positive integer n. Now, we extend the concept to multigroup of G as follows.

Definition 12. Let A and B be multigroups of G such that $A \subseteq B$. Then A is a pure multigroup of G if $nA = A \cap nB$ for every $n \in \mathbb{N}$. Equivalently, $C_{nA}(x) = C_A(x) \wedge C_{nB}(x) \,\forall x \in G$ for every $n \in \mathbb{N}$.

Remark 2. Suppose A, B and C are multigroups of G such that A and B are contained in C. If A and B are pure, then $A \cap B$ and $A \cup B$ are pure multigroups of G since $n(A \cap B) = (A \cap B) \cap nC$ and $n(A \cup B) = (A \cup B) \cap nC$.

Definition 13. Let A and B be multigroups of G such that $A \subseteq B$. Then A is p-pure if $p^r A = A \cap p^r B$ for every $r \in \mathbb{N}$ and p is prime. Equivalently, if $C_{p^r A}(x) = C_A(x) \wedge C_{p^r B}(x) \,\forall x \in G$ for every $r \in \mathbb{N}$ and p is prime.

   In fact, every p-pure multigroup of a p-divisible group is p-divisible.

Proposition 7. *Suppose $f: G \to G'$ is an isomorphism of divisible groups, A, C are multigroups of G and B, D are multigroups of G' such that $A \subseteq C$ and $B \subseteq D$. If A and B are pure multigroups of G and G', respectively then*

   (i) $f(A)$ is a pure multigroup of G',
   (ii) $f^{-1}(B)$ is a pure multigroup of G.

Proof. Let $x, y \in G$ and $w, z \in G'$ such that $f(x, y) = w, z$. Since G, G' are divisible groups for every $n \in \mathbb{N}$, $\exists\, y \in G$ and $z \in G'$ such that $ny = x$, $nz = w$. From Proposition 2, it follows that $f(A)$ and $f^{-1}(B)$ are multigroups of G and G', respectively. Assume that C and D are constant, then by Theorems 3 and 4, we deduce that, for any $w \in G'$

$$
\begin{aligned}
C_{f(nA)}(w) \quad &= \quad C_{nf(A)}(w) = C_{f(A)}(w) \wedge C_{nf(C)}(w) \\
&= \quad C_{f(A)}(w) \wedge \bigvee_{w=nz} C_{f(C)}(z) \\
&= \quad C_{f(A)}(w)
\end{aligned}
$$

and

$$
\begin{aligned}
C_{f^{-1}(nB)}(x) &= C_{nf^{-1}(B)}(x) = C_{f^{-1}(B)}(x) \wedge C_{nf^{-1}(D)}(x) \\
&= C_{f^{-1}(B)}(x) \wedge \bigvee_{x=ny} C_{f^{-1}(D)}(y) \\
&= C_{f^{-1}(B)}(x).
\end{aligned}
$$

Hence, the results.
$\square$

Theorem 11. *Let $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ be families of multigroups of G such that $\{A_i\}_{i \in I} \subseteq \{B_j\}_{j \in J}$ and $\{B_j\}_{j \in J}$ is constant. If $\{A_i\}_{i \in I}$ is pure, then $\bigcap_{i \in I} A_i$ is a pure multigroup of G.*

Proof. Certainly, $\bigcap_{i \in I} A_i$ is a multigroup of G by Proposition 1. If every A_i is pure, then suppose $x \in nG$, we have

$$
\begin{aligned}
C_{n(\bigcap_{i \in I} A_i)}(x) &= C_{(\bigcap_{i \in I} A_i) \cap n(\bigcap_{j \in J} B_j)}(x) \\
&= \bigwedge_{i \in I} CM_{A_i}(x) \wedge \bigvee_{x=ny} \bigwedge_{j \in J} C_{B_j}(y) \\
&= \bigwedge_{i \in I} C_{A_i}(x) \wedge \bigwedge_{j \in J} \bigvee_{x=ny} C_{B_j}(y) \\
&= \bigwedge_{i \in I} C_{A_i}(x) = C_{\bigcap_{i \in I} A_i}(x).
\end{aligned}
$$

Again, suppose $x \notin nG$, then

$$
C_{n(\bigcap_{i \in I} A_i)}(x) = C_{(\bigcap_{i \in I} A_i) \cap n(\bigcap_{j \in J} B_j)}(x) = 0 = C_{\bigcap_{i \in I} A_i}(x).
$$

Hence, $\bigcap_{i \in I} A_i$ is a pure multigroup of G. $\square$

Theorem 12. *Let $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ be families of multigroups of G such that $\{B_j\}_{j \in J}$ is constant and contains $\{A_i\}_{i \in I}$. If $\{A_i\}_{i \in I}$ is pure, then*

(i) *$\bigcup_{i \in I} A_i$ is a pure multigroup of G for $A_1 \subseteq A_2 \subseteq ... \subseteq A_n$, $i = 1,...,n$.*
(ii) *$\Sigma_{i \in I} A_i$ is a pure multigroup of G.*

Proof. By synthesizing Proposition 1 and following the same argument in Theorem 11, the proofs of (i) and (ii) are established. $\square$

Theorem 13. *Suppose A and B are multigroups of G such that $A \subseteq B$ and B is constant. Then A is pure if and only if A is a divisible multigroup of G.*

Proof. Suppose A is a pure multigroup of G. Then $nA = A \cap nB$ for $n \in \mathbb{N}$. Certainly, $A \cap nB = A$ because

$$
\begin{aligned}
C_A(x) \wedge C_{nB}(x) &= C_A(x) \wedge \bigvee_{ny=x} C_B(y) \\
&= C_A(x) \ \forall x \in X.
\end{aligned}
$$

Thus, $nA = A$ for $n \in \mathbb{N}$, and hence A is a divisible multigroup of G.

Conversely, assume A is a divisible multigroup of G. Then $nA = A$ for $n \in \mathbb{N}$. Since $A \cap nB = A$, we have $nA = A \cap nB$ $n \in \mathbb{N}$, and so A is a pure multigroup of G. $\square$

Proposition 8. *Suppose A and B be multigroups of G such that $A \subseteq B$ and B is constant. Then A is pure if and only if it is p-pure for every prime p.*

Proof. Similar to Proposition 6. $\square$

Proposition 9. *Let A and B be multigroups of G such that $A \subseteq B$ and B is constant. If A is a pure multigroup of G, then A_n, $n \in \mathbb{N}$ is a pure subgroup of G for $n \leq C_A(0)$ and A^n, $n \in \mathbb{N}$ is a pure subgroup of G for $n \geq C_A(0)$.*

Proof. Similar to Theorem 5. □

Proposition 10. *Suppose A and B are multisets of G bounded by G^n such that $A \subseteq B$ and B is constant. If $C_A(0) = n$ and every A_n, $n \in \mathbb{N}$, is a pure subgroup of G, then A is a pure multigroup of G.*

Proof. Similar to Theorem 6. □

Corollary 14. *Let A and B be multisets of G bounded by G^n such that $A \subseteq B$ and B is constant. If $C_A(0) = n$ and every A^n, $n \in \mathbb{N}$, is a pure subgroup of G, then A is a pure multigroup of G.*

Proof. Similar to Corollary 7. □

4. Conclusion

Multigroup theory is an application of group theory to multisets. Numerous analog ideas of group theory have been extended to multigroups. In this paper, we have introduced the concepts of divisible and pure multigroups with a number of some related results. The interconnection between divisible and pure multigroups and that of divisible and pure groups were established with the aid of multigroup cuts. Additional characterizations of divisible and pure fuzzy multigroups could be explored in future research.

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