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On G^{β}-Property of G-Metric Spaces

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This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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ABSTRACT

The purpose of this paper is to introduce and investigate weak form of G-open sets in G-metric spaces, namely G^{β}-open sets. The relationships among this form with the other known sets are introduced. We give the notions of the interior operator, the closure operator and frontier operator via G^{β}-open sets.

Keywords: Open set; Metric spaces.

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1 INTRODUCTION

The concept of a metric space was introduced by Frechet in 1906, [1]. It has a very important basic role in mathematics and its application. Many mathematical concepts that can be discussed in this space.The first attempt to generalize the ordinary distance function to a distance of three points was introduced by Gahler, [2, 3], in 1993. K. S. Ha, et al; [4], showed that a 2-metric is not a generalization of the usual notion of a metric. It was mentioned by Gahler, [2], that the notion of a 2-metric is an extension of an idea of ordinary metric and geometrically (x, y, z) represents the area of a triangle formed by the points x,y and z in X as its vertices. But this is not always true.A.Sharma, [5], showed that $(x, y, z) = 0$ for any three distinct points $x, y, z \in R^2$. B. C. Dhage

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in 1963 introduced a new class of generalized metrics called D-metrics, [3],. However, several errors for fundamental topological properties in a D-metric space were found by Z.Mustafa and B.Sims, [6]. Due to these considerations, Z. Mustafa and B.Sims , [7], proposed a more appropriate notion of a generalized metric space, called G-metric space.

This paper is organized as follows. Section 2 is devoted to some preliminaries. Section 3 introduces the concept of G^{β}-open sets by utilizing the G-open balls. Furthermore, the relationship with the other known sets will be studied. In Section 4 we introduce the concepts of the interior operator, the closure operator and frontier operator via G^{β}-open sets.

2 PRELIMINARIES

Definition 2.1. *[1] Let X be any nonempty set. A function $d : X \times X \to [0, \infty)$ is called a metric function on X if it satisfies the following three conditions for all $x, y, z \in X$:*

1. *(positive property) $d(x, y) \geq 0$ with equality if and only if $x = y$;*
2. *(symmetric property) $d(x, y) = d(x, y)$;*
3. *(triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.*

A pair (X, d), where d is a metric on X is called a metric space.

Definition 2.2. *[6] Let X be a nonempty set and $\mathbb{R}$ be the set of real numbers. A function G : $X \times X \times X \to R$ is called a G-metric function on X if it satisfies the following:*

1. *$G(x, x, y) > 0$ for all $x \neq y \in X$;*
2. *$G(x, y, z) = 0$ if and only if $x = y = z$;*
3. *$G(x, x, y) \leq G(x, y, z)$ for every $x, y, z \in X$ with $y \neq z$;*
4. *$G(x, y, z) = G(p(x, y, z))$ for every $x, y, z \in X$and for any permutation p of x, y, z;*
5. *$G(x, y, z) \leq G(x, u, u) + G(u, y, z)$ for every $x, y, z, u \in X$.*

If G is a G-metric function on X, then the pair (X, G) is called a G-metric space.

Example 2.3. *[7] Let $(\mathbb{R}, d)$ be the usual metric space. Define G_s by $G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z)$ for all $x, y, z \in \mathbb{R}$. Then it is clear that $(\mathbb{R}, G_s)$ is a G-metric space.*

Example 2.4. *[7] Let $X = \{a, b\}$. Define G on $X \times X \times X$ by G(a,a,a) = G(b,b,b) = 0, G(a a,b) = 1,G(a,b,b) = 2.*

Example 2.5. *[7] Let $(\mathbb{R}, G)$ be G-metric space defined by $G(x, y, z) = max\{|x - y|, |y - z|, |z - x|\}$.*

Definition 2.6. *[8] Let (X, G) be a G-metric space, $x \in X$ and $A \subseteq X$. The open ball with center x and radius ϵ in metric space (X, G) is denoted by $B_G(x, \epsilon)$ and defined by*

$$B_G(x, \epsilon) = \{y \in X | d(x, y, y) < \epsilon\}.$$

The closed ball with center x and radius ϵ in G-metric space (X, G) is denoted by $C_G(x, \epsilon)$and defined by

$$C_G(x, \epsilon) = \{y \in X | d(x, y, y) \leq \epsilon\}.$$

The set A is called an open set in G-metric space (X, G) if for every $x \in A$, there is $\epsilon > 0$ such that $B_G(x, \epsilon) \subseteq A$. The set A is called closed set in metric space (X, G) if $X - A$ is an open set in G-metric space (X, G).

Theorem 2.7. *[8] Every G-open ball $B_G(x, \epsilon), x \in X, \epsilon > 0$ is an open set in X.*

Theorem 2.8. *[7] Let (X, G) be a G-metric space, then for any $x \in X$ and $\epsilon > 0$, we have.*
(1) If $G(y, x, x) < \epsilon$ then $x, y \in B_G(x, \epsilon)$;
(2) If $y \in B_G(x, \epsilon)$ then there exists a $\delta > 0$ such that $B_G(y, \delta) \subseteq B_G(x, \epsilon)$.

Definition 2.9. *[8] $Cl_G(A)$is called the G-closure of A if it is the intersection of all G-closed sets containing A.*

Definition 2.10. *[8] A set U in a G-metric space X, is said to be closed if its complement X - U is G-open.*

3 G^β-OPEN SETS

Definition 3.1. *Let (X, G) be a G-metric space and $A \subseteq X$.A point $x \in X$ is called a G-point of A in G-metric space (X, G) if there is $\delta > 0$ such that for every $y \in B_G(x, \delta)$,*

$$B_G(y, \epsilon) \cap G \neq \emptyset \quad \forall \epsilon > 0.$$

$G^\beta(A)$ denotes the set of all G^β-points of A in G-metric space (X, G)

Example 3.2. *Let $(\mathbb{R}, G)$ be G-metric space defined by $G(x, y, z) = max\{|x-y|, |y-z|, |z-x|\}$. Let $A = (0, 2)$ and $B = Q$ be that set of rational numbers. Note that $G^\beta(A) = (0, 2)$ and $G^\beta(B) = \mathbb{R}$.*

Theorem 3.3. *Let (X, G) be any G-metric space and $A, B \subseteq X$.Then*

1. *$G^\beta(\phi) = \phi$ and $G^\beta(X) = X$;*
2. *if $A \subseteq B$ Then $G^\beta(A) \subseteq G^\beta(B)$;*
3. *$G^\beta(A \cap B) \subseteq G^\beta(A) \cap G^\beta(B)$;*
4. *$G^\beta(A) \cup G^\beta(B) \subseteq G^\beta(A \cup B)$.*

Proof.
1. It is clear from the definition ,we get that $G^\beta(\phi) = \phi$ and $G^\beta(X) = X$.
2. Let $A \subseteq B$ and $x \in G^\beta(A)$. Then is $\delta > 0$ such that for every $y \in B_G(y, \epsilon) \cap A \neq \emptyset$, for all Since $A \subseteq B$. Then $B_G(y, \epsilon) \cap B \neq \emptyset$, for all $\epsilon > 0$. That is, $x \in G^\beta(B)$. Then $G^\beta(A) \subseteq G^\beta(B)$.
3. Since $A \cap B \subseteq A$. Then by part (2) $G^\beta(A \cap B) \subseteq G^\beta(A)$. Similar $G^\beta(A \cap B) \subseteq G^\beta(B)$ Then $G^\beta(A \cap B) \subseteq G^\beta(A) \cap G^\beta(B)$.
4. Since $A \subseteq (A \cup B)$. Then by part (2) $G^\beta(A) \subseteq G^\beta(A \cup B)$. Similar $G^\beta(B) \subseteq G^\beta(A \cup B)$ Then $G^\beta(A) \cup G^\beta(B) \subseteq G^\beta(A \cup B)$.

□

Definition 3.4. *Let (X, G) be a G-metric space. A subset $A \subseteq X$ is called a G^β-open set in G-metric space (X, G) if for every $x \in A$,*

$$B_G(x, \epsilon) \cap \ G^\beta(A) \neq \emptyset \quad \forall \epsilon > 0.$$

A subset $A \in X$ is called a G^β-closed set in G-metric space (X, G) if $X - A$ is a G^β-open set in G-metric space (X, G).

Example 3.5. *In Example(3.2), the sets A and B are G^β-open sets. Note that any finite sub sets of $\mathbb{R}$ are not G^β-open set.*

Theorem 3.6. *Every G-open set is a G^β-open set.*

Proof. Let A be any G-open set in G-metric space (X, G). Let $x \in A$ be arbitrary point. Then there is $\delta > 0$ such that $B_G(x, \varepsilon) \subseteq G$. For every $y \in B_G(x, \varepsilon)$, $y \in B_G(x, \varepsilon)(y)$ and $y \in A$ for every $\varepsilon > 0$. That is, $B_G(y, \varepsilon) \cap G \neq \emptyset$ for every $\varepsilon > 0$. Hence A is G^β-open set. □

The converse of above theorem need not be true.

Example 3.7. *In Example(3.2), note that for the closed interval $A = [a, b]$, $G^\beta(A) = (a, b)$. Then it is clear to check that A is a G^β-open set. Take $x = a$ or $x = b$. Note that $x \in A$ but there is no G-open ball with center x contained in A. That is, A is not G-open set in $(\mathbb{R}, G)$.*

The intersection of two G^β-open sets no need to be G^β-open set. In Example(3.2), set o f rational numbers Q is a G^β-open set but not G-open set in $(\mathbb{R}, G)$ and the set $IR \cup \{q\}$ is a G^β-open set in $(\mathbb{R}, G)$, where IR is the set of irrational numbers and q is any rational number, but $Q \cap (IR \cup \{q\}) = \{q\}$ is not G^β-open set. That is, the collection of all G^β-open sets in G-metric space (X, G) does not form topology on a set X.

The following theorem shows that the intersection of a G-open set and a G^β-open set is a G^β-open set.

Theorem 3.8. *The intersection of a G-open set and a G^β-open set is a G^β-open set.*

Proof. Let A be G-open set and B be G^β-open set in G-metric space in (X, G). Let $x \in A \cap B$ be arbitrary point. Then $x \in A$ and $x \in B$. Then there are $\delta_1 > 0$ and $\delta_2 > 0$ such that $B_G(x, \delta_1) \subseteq A$ and for every $y \in B_G(x, \delta_2)$, $B_G(y, \varepsilon) \cap B \neq \emptyset$ for every $\varepsilon > 0$. Take $\delta = min\{\delta_1, \delta_2\} > 0$. Then $B_G(x, \delta) \subseteq A$ and for every $y \in B_G(x, \delta)$, $B_G(y, \varepsilon)) \cap B \neq \emptyset$ for every $\varepsilon > 0$. Now for every $y \in B_G(x, \delta)$ and since A is G-open set, then there is $\varepsilon_y > 0$ such that $B_G(y, \varepsilon_y)) \subseteq A$ and $B_G(y, min\{\delta_1, \delta_2\}) \cap B \neq \emptyset$. Since $B_G(y, min\{\delta_1, \delta_2\}) \cap B \subseteq B_G(y, \varepsilon)) \cap A \cap B$, then $B_G(y, \varepsilon) \cap (A \cap B) \neq \emptyset$ for every $\varepsilon > 0$. That is $A \cap B$ is G-open set. Hence $x \in G^\beta(A \cap B)$. Then $B_G(y, \varepsilon) \cap G^\beta(A \cap B) \neq \emptyset$ for all $\varepsilon > 0$. There for $A \cap B$ is G^β-open set. □

Theorem 3.9. *The union of any family of G^β-open sets is G^β-open set.*

Proof. Let H_λ be a G^β-open in G-metric space (X, G) for all $\lambda \in \Delta$. Let $x \in \cup_{\lambda \in \Delta} H_\lambda$ be an arbitrary point. Then there is at least $\lambda_0 \in \Delta$ such that $x \in H_{\lambda_0}$. Since H_{λ_0} is a G^β-open set then $B_G(x, \varepsilon) \cap G^\beta(H_{\lambda_0}) \neq \emptyset$ for all $\varepsilon > 0$. Hence by Theorem (3.3), $G^\beta(H_{\lambda_0}) \subseteq G^\beta(\cup_{\lambda \in \Delta})$. Hence $B_G(x, \varepsilon) \cap G^\beta(\cup_{\lambda \in \Delta} H_\lambda) \neq \emptyset$ for all $\varepsilon > 0$. That is $\cup_{\lambda \in \Delta} H_\lambda$ is G^β-open set.

□

4 G^β-OPEN OPERATORS

In this section, we define the interior operator, the closure operator and frontier operator via G^β-open sets.

Definition 4.1. Let (X, G) be a G-metric space and $A \subseteq X$. The G-closure operator of A is denoted by $Cl_G^\beta(A)$ and defined by

$$Cl_G^\beta(A) = \cap\{H \subseteq X : A \subseteq H \text{ and } H \text{ is } G^\beta\text{-closed set}\}.$$

The G-interior functor of A is denoted by $Int_G^\beta(A)$ and defined by

$$Int_G^\beta(A) = \cup\{H \subseteq X : H \subseteq A \text{ and } H \text{ is } G^\beta\text{-open set}\}.$$

Remark 4.2.

1. By Theorem(3.9), $Cl^{\beta}_{G}(A)$ is a G^{β}-closed set and $Int^{\beta}_{G}(A)$ is G^{β}-open set in G-metric space (X, G).
2. For a G-metric space (X, G) and $A \subseteq X$, it is clear from the definition of $Cl^{\beta}_{G}(A)$ and $Int^{\beta}_{G}(A)$ that $A \subseteq Cl^{\beta}_{G}(A)$ and $Int^{\beta}_{G}(A) \subseteq A$.

Theorem 4.3. For a G-metric space (X, G) and $A \subseteq X$, $Cl^{\beta}_{G}(A) = A$ if and only if A is a G^{β}-closed set.

Proof. Let $Cl^{\beta}_{G}(A) = A$. Then from definition of $Cl^{\beta}_{G}(A)$ and Theorem(3.9), $Cl^{\beta}_{G}(A)$ is a G^{β}-closed set and A is a G^{β}-closed set. Conversely, we have $A \subseteq Cl^{\beta}_{G}(A)$ by Remark(4.2). Since A is a G^{β}-closed set, then it is clear from the definition of $Cl^{\beta}_{G}(A)$, $Cl^{\beta}_{G}(A) \subseteq A$. Hence $A = Cl^{\beta}_{G}(A)$. □

Theorem 4.4. For a G-metric space (X, G) and $A \subseteq X$, and $Int^{\beta}_{G}(A) = A$ if and only if A is a G^{β}-open set.

Proof. Let A be G^{β}-open set. Then for all $x \in A$, we have $x \in A \subseteq A$. That is, $A \subseteq Int^{\beta}_{G}(A)$. Then $A = Int^{\beta}_{G}(A)$ from Remark(4.2). The converse is trivial. □

Theorem 4.5. For a G-metric space (X, G) and $A \subseteq X$, $x \in Cl^{\beta}_{G}(A)$ if and only if for all G^{β}-open set B containing x, $B \cap A \neq \emptyset$.

Proof. Let $x \in Cl^{\beta}_{G}(A)$ and B be any G^{β}-open set containing x. If $B \cap A = \emptyset$ then $A \subseteq X - B$. Since $X - B$ is a G^{β}-closed set containing A, then $Cl^{\beta}_{G}(A) \subseteq X - B$ and so $x \in Cl^{\beta}_{G}(A) \subseteq X - B$. Hence this is contradiction, because $x \in B$. Therefore $B \cap A \neq \emptyset$.

Conversely, Let $x \notin Cl^{\beta}_{G}(A)$. Then $X - Cl^{\beta}_{G}(A)$ is a G-open set containing x. Hence by hypothesis, $[X - Cl^{\beta}_{G}(A)] \cap A \neq \emptyset$. But this is contradiction, because $X - Cl^{\beta}_{G}(A) \subseteq X - A$. □

Theorem 4.6. For a G-metric space (X, G) and $A \subseteq X$, $x \in Int^{\beta}_{G}(A)$ if and only if there is G^{β}-open set B such that $x \in B \subseteq A$.

Proof. Let $x \in Int^{\beta}_{G}(A)$ and take $B = Int^{\beta}_{G}(A)$. Then by Theorem(4.5) and definition of $Int^{\beta}_{G}(A)$ we get that B is a G^{β}-open set and by Remark(4.2), $x \in B \subseteq A$. Conversely, let there is G^{β}-open set B such that $x \in B \subseteq A$ Then by definition of $Int^{\beta}_{G}(A)$, $x \in B \subseteq Int^{\beta}_{G}(A)$. □

Theorem 4.7. For a G-metric space (X, G) and $A, B \subseteq X$, the following hold:

1. If $A \subseteq B$ then $Cl^{\beta}_{G}(A) \subseteq Cl^{\beta}_{G}(B)$;
2. $Cl^{\beta}_{G}(A) \cup Cl^{\beta}_{G}(B) \subseteq Cl^{\beta}_{G}(A \cup B)$;
3. $Cl^{\beta}_{G}(A \cap B) \subseteq Cl^{\beta}_{G}(A) \cap Cl^{\beta}_{G}(B)$;
4. $Cl^{\beta}_{G}(A) \subseteq Cl_{G}(A)$.

Proof.

1. Let $x \in Cl^{\beta}_{G}(A)$. Then by Theorem(4.5), for all G^{β}-open set C containing x, $C \cap A \neq \emptyset$. Since $A \subseteq B$ then $C \cap B \neq \emptyset$. Hence $x \in Cl^{\beta}_{G}(B)$. That is, $Cl^{\beta}_{G}(A) \subseteq Cl^{\beta}_{G}(B)$.
2. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then by part(1), $Cl^{\beta}_{G}(A) \subseteq Cl^{\beta}_{G}(A \cup B)$ and $Cl^{\beta}_{G}(B) \subseteq Cl^{\beta}_{G}(G \cup B)$. Hence $Cl^{\beta}_{G}(G) \cup Cl^{\beta}_{G}(B) \subseteq Cl^{\beta}_{G}(A \cup B)$.
3. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then by part(1), $Cl^{\beta}_{G}(A \cap B) \subseteq Cl^{\beta}_{G}(A)$ and $Cl^{\beta}_{G}(A \cap B) \subseteq Cl^{\beta}_{G}(B)$. Hence $Cl^{\beta}_{G}(A \cap B) \subseteq Cl^{\beta}_{G}(A) \cap Cl^{\beta}_{G}(B)$.
4. It is clear from Theorem(4.5) and from every G-open set is G^{β}-open set.

□

In the above theorem $Cl^{\beta}_{G}(A \cup B) \neq Cl^{\beta}_{G}(A) \cup Cl^{\beta}_{G}(B)$ as it is shown in the following example.

Example 4.8. Let $(\mathbb{R}, G)$ be G-metric space, where

$$G(x, y, z) = max\{|x-y|, |y-z|, |z-x|\}$$

and $(\mathbb{R}, d)$ is usual metric space. Let $A = IR$ and $B = Q - [\{2\}]$, where Q is the set of rational numbers, IR is the set of irrational numbers and 2 is any rational number. Since A and B are G^β-closed sets in $\mathbb{R}$. Then $Cl^\beta_G(A) \cup Cl^\beta_G(B) = A \cup B = \mathbb{R} - \{2\}$. If $\mathbb{R} - \{2\}$ is G^β-closed set in $\mathbb{R}$ then $\{2\}$ is G^β-open set but $\{2\}$ is not G^β-open set and this contradiction. Hence $\mathbb{R} - \{2\}$ is not G^β-closed set in $\mathbb{R}$. Since $\mathbb{R} - \{2\} \subseteq Cl^\beta_G(\mathbb{R} - \{2\})$ then

$$Cl^\beta_G(A \cup B) = Cl^\beta_G(\mathbb{R} - \{2\}) = \mathbb{R}.$$

Theorem 4.9. For a G-metric space (X, G) and $A, B \subseteq X$, the following hold:

1. If $A \subseteq B$ then $Int^\beta_G(A) \subseteq Int^\beta_G(B)$;
2. $Int^\beta_G(A) \cup Int^\beta_G(B) \subseteq Int^\beta_G(A \cup B)$;
3. $Int^\beta_G(A \cap B) \subseteq Int^\beta_G(B) \cap Int^\beta_G(B)$;
4. $Int_G(A) \subseteq Int^\beta_G(A)$.

Proof.
1. Let $x \in Int^\beta_G(A)$. Then by Theorem(4.6), there is G^β-open set C such that $x \in C \subseteq A$ Since $A \subseteq B$ then $x \in C \subseteq B$. Hence $x \in Int^\beta_G(B)$. That is, $Int^\beta_G(A) \subseteq Int^\beta_G(B)$.
2. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then by part(1), $Int^\beta_G(A) \subseteq Int^\beta_G(A \cup B)$ and $Int^\beta_G(B) \subseteq Int^\beta_G(A \cup B)$. Hence $Cl^\beta_G(A) \cup Int^\beta_G(B) \subseteq Int^\beta_G(A \cup B)$.
3. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then by part(1), $Int^\beta_G(A \cap B) \subseteq Int^\beta_G(A)$ and $Int^\beta_G(A \cap B) \subseteq Int^\beta_G(B)$. Hence $Int^\beta_G(A \cap B) \subseteq Int^\beta_G(A) \cap Int^\beta_G(B)$.
4. It is clear from Theorem(4.5) and from every G-open set is G^β-open set.

□

In the last theorem $Int^\beta_G(A \cap B) \neq Int^\beta_G(A) \cap Int^\beta_G(B)$ as it is shown in the following example.

Example 4.10. In Example(4.8), take $A = Q \cup \{\sqrt{2}\}$ and $B = IR$, where Q is the set of rational numbers, IR is the set of irrational numbers and $\sqrt{2}$ is any irrational number. Since A and B are G^β-open sets in $\mathbb{R}$. Then $Int^\beta_G(A) \cap Int^\beta_G(B) = A \cap B = (Q \cup \{\sqrt{2}\}) \cap IR = \{\sqrt{2}\}$. Since $\{\sqrt{2}\}$ is not G^β-open set and $Int^\beta_G(\{\sqrt{2}\}) \subseteq \{\sqrt{2}\}$ then $Int^\beta_G(A \cap B) = Int^\beta_G(\{\sqrt{2}\}) = \emptyset$.

Theorem 4.11. For a G-metric space (X, G) and $G \subseteq X$, the following hold:

1. $Int^\beta_G(X - A) = X - Cl^\beta_G(A)$;
2. $Cl^\beta_G(X - A) = X - Int^\beta_G(A)$.

Proof.
1. Since $A \subseteq Cl^\beta_G(A)$, then $X - Cl^\beta_G(A) \subseteq X - A$. Since $Cl^\beta_G(A)$ is a G^β-closed set then $X - Cl^\beta_G(A)$ is a G-open set. Then

$$X - Cl^\beta_G(A) = Int^\beta_G[X - Cl^\beta_G(A)] \subseteq Int^\beta_G(X - A).$$

For the other side, let $x \in Int^\beta_G(X - A)$. Then there is G^β-open set C such that $x \in C \subseteq X - A$. Then $X - C$ is a G^β-closed set containing A and $x \notin X - C$. Hence $x \notin Cl^\beta_G(G)$, that is, $x \in X - Cl^\beta_G(A)$.

2. Since $Int^\beta_G(A) \subseteq A$, then $X - A \subseteq X - Int^\beta_G(A)$. Since $Int^\beta_G(A)$ is a G^β-open set then $X - Int^\beta_G(A))$ is a G^β-closed set. Then

$$Cl^\beta_G(X - A) = Cl^\beta_G[X - Int^\beta_G(A)] = X - Int^\beta_G(A).$$

For the other side, let $x \notin Cl^\beta_G(X - A)$. Then by Theorem(4.5), there is a G^β-open set C containing x such that $C \cap (X - A) = \emptyset$. Then $x \in C \subseteq A$, that is, $x \in Int^\beta_G(A)$. Hence $x \notin X - Int^\beta_G(A)$. Therefore $X - Int^\beta_G(A) \subseteq Cl^\beta_G(X - A)$.

□

Theorem 4.12. For a subset $A \subseteq X$ of G-metric space (X, G) the following hold:

1. If B is a G-open set in X then $Cl_G^\beta(A) \cap B \subseteq Cl_G^\beta(A \cap B)$;
2. If B is a G-closed set in X then $Int_G^\beta(A \cup B) \subseteq Int_G^\beta(A) \cup B$.

Proof. 1. Let $x \in Cl_G^\beta(A) \cap B$. Then $x \in Cl_G^\beta(A)$ and $x \in B$. Let D be any G^β-open set in (X, G) containing x. By Theorem(3.8), $D \cap B$ is G^β-open set containing x. Since $x \in Cl_G^\beta(A)$ then by Theorem(4.5), $(D \cap B) \cap A \neq \emptyset$. This implies, $D \cap (B \cap A) \neq \emptyset$. Hence by Theorem(4.5), $x \in Cl_G^\beta(A \cap B)$. That is, $Cl_G^\beta(A) \cap B \subseteq Cl_G^\beta(A \cap B)$.

2. Since B is a G-closed set X then by the part(1) and Theorem(4.11),

$$\begin{aligned} X - [Int_G^\beta(A) \cup B] &= [X - Int_G^\beta(A)] \cap [X - B] \\ &= [Cl_G^\beta(X - A)] \cap [X - B] \\ &\subseteq Cl_G^\beta[(X - A) \cap (X - B)] \\ &= Cl_G^\beta(X - (A \cup B)) \\ &= X - (Int_G^\beta(A \cup B)). \end{aligned}$$

Hence $Int_G^\beta(A \cup B) \subseteq Int_G^\beta(A) \cup B$.

□

Theorem 4.13. For a G-metric space (X, G) and $A \subseteq X$, $x \in Cl_G(A)$ if and only if for all $\varepsilon > 0$, $B_G(x, \varepsilon) \cap A \neq \emptyset$.

Proof. Let $x \in Cl_G(A)$ and $\varepsilon > 0$. If $B_G(x, \varepsilon) \cap A = \emptyset$ then $A \subseteq X - B_G(x, \varepsilon)$. Since $X - B_G(x, \varepsilon)$ is a G-closed set containing A, then $Cl_G(A) \subseteq X - B_G(x, \varepsilon)$ and $x \in Cl_G(A) \subseteq X - B_G(x, \varepsilon)$. Hence this is contradiction, because $x \in B_G(x, \varepsilon)$. Therefore $B_G(x, \varepsilon) \cap A \neq \emptyset$.

Conversely, Let $x \notin Cl_G(A)$. Then $X - Cl_G(A)$ is a G-open set containing x. Then there is $\varepsilon > 0$ such that $B_G(x, \varepsilon) \subseteq X - Cl_G(A)$ Hence by hypothesis, $B_G(x, \varepsilon) \cap A \neq \emptyset$. But this is contradiction, because $B_G(x, \varepsilon) \subseteq X - Cl_G(A) \subseteq X - A$. □

For a subset A of G-metric space (X, G) the G-frontier operator of A is defined by

$$\Gamma_G^\beta(A) = Cl_G^\beta(A) - Int_G^\beta(A).$$

Theorem 4.14. For a subset $A \subseteq X$ of G-metric space (X, G), the following hold:

1. $Cl_G^\beta(A) = \Gamma_G^\beta(A) \cup Int_G^\beta(A)$;
2. $\Gamma_G^\beta(A) \cap Int_G^\beta(A) = \emptyset$;
3. $\Gamma_G^\beta(A) = Cl_G^\beta(A) \cap Cl_G^\beta(X - A)$.

Proof. 1. Note that

$$\begin{aligned} \Gamma_G^\beta(A) \cup Int_G^\beta(A) &= (Cl_G^\beta(A) - Int_G^\beta(A)) \cup Int_G^\beta(A) \\ &= [Cl_G^\beta(A) \cap (X - Int_G^\beta(A))] \cup Int_G^\beta(A) \\ &= [Cl_G^\beta(A) \cup Int_G^\beta(A)] \cap [(X - Int_G^\beta(A)) \cup Int_G^\beta(A)] \\ &= Cl_G^\beta(A) \cap X = Cl_G^\beta(A). \end{aligned}$$

2. It is clear from the definition of $\Gamma_G^\beta(A)$.
3. By Theorem(4.11),

$$\begin{aligned} \Gamma_G^\beta(A) &= Cl_G^\beta(A) - Int_G^\beta(A) = Cl_G^\beta(A) \cap (X - Int_G^\beta(A)) \\ &= Cl_G^\beta(A) \cap Cl_G^\beta(X - A). \end{aligned}$$

□

Corollary 4.15. For a subset $A \subseteq X$ of G-metric space (X, G), $\Gamma^{\beta}_{G}(A)$ is G^{β}-closed set in (X, G).

Proof. By Theorem(4.9) and the part(3) of the last theorem. □

Theorem 4.16. For a subset $A \subseteq X$ of G-metric space (X, G), the following hold:

1. A is a G^{β}-open set if and only if $\Gamma^{\beta}_{G}(A) \cap A = \emptyset$;

2. A is a G^{β}-closed set if and only if $\Gamma^{\beta}_{G}(A) \subseteq A$;

3. A is both G^{β}-open set and G^{β}-closed set if and only if $\Gamma^{\beta}_{G}(A) = \emptyset$.

Proof.

1. Let A be a G^{β}-open set. Then $Int^{\beta}_{G}(A) = A$. Then by Theorem(4.14),

$$\Gamma^{\beta}_{G}(A) \cap A = \Gamma^{\beta}_{G}(A) \cap Int^{\beta}_{G}(A) = \emptyset$$

Conversely, suppose that $\Gamma^{\beta}_{G}(A) \cap A = \emptyset$. Then

$$\begin{aligned} A - Int^{\beta}_{G}(A) &= [A \cap Cl^{\beta}_{G}(A)] - [A \cap Int^{\beta}_{G}(A)] \\ &= A \cap (Cl^{\beta}_{G}(A) - Int^{\beta}_{G}(A)) = A \cap \Gamma^{\beta}_{G}(A) = \emptyset. \end{aligned}$$

That is, $Int^{\beta}_{G}(A) = A$. Hence A is a G^{β}-open set.

2. Let A be a G^{β}-closed set. Then $Cl^{\beta}_{G}(A) = A$. Then

$$\Gamma^{\beta}_{G}(A) = Cl^{\beta}_{G}(A) - Int^{\beta}_{G}(A) = A - Int^{\beta}_{G}(A) \subseteq A.$$

Conversely, suppose that $\Gamma^{\beta}_{G}(A) \subseteq A$. Then by Theorem(4.14),

$$Cl^{\beta}_{G}(A) = Int^{\beta}_{G}(A) \cup \Gamma^{\beta}_{G}(A) \subseteq Int^{\beta}_{G}(A) \cup A \subseteq A.$$

That is, $Cl^{\beta}_{G}(A) = A$. Hence A is G^{β}-closed set.

3. Let A be both G^{β}-closed set and G^{β}-open set. Then $Cl^{\beta}_{G}(A) = A = Int^{\beta}_{G}(A)$. Then

$$\Gamma^{\beta}_{G}(A) = Cl^{\beta}_{G}(A) - Int^{\beta}_{G}(A) = A - A = \emptyset.$$

Conversely, suppose that $\Gamma^{\beta}_{G}(A) = \emptyset$. Then $Cl^{\beta}_{G}(A) - Int^{\beta}_{G}(A) = \emptyset$. Since $Int^{\beta}_{G}(A) \subseteq Cl^{\beta}_{G}(A)$ then $Cl^{\beta}_{G}(A) = Int^{\beta}_{G}(A)$. Since $Int^{\beta}_{G}(A) \subseteq A \subseteq Cl^{\beta}_{A}(A)$ then

$$Cl^{\beta}_{G}(A) = A = Int^{\beta}_{G}(A).$$

That is, $Cl^{\beta}_{G}(A) = A$. Hence A is both G^{β}-closed set and G^{β}-open set.

□

5 CONCLUSION

As we noted that the G^{β}-open set is a weak form of open set in G-metric space, also the reader can give the notion of the continty property via G^{β}-open sets in G-metric spaces.The reader also can introduce sepertion axioms connectedness and compactness properties by using G^{β}-open sets in G-metric spaces.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

[1] Frechet M. Sur quelques points du calcul fonctionnel. Rendiconti del Circolo

Matematico di Palermo. 1906;22(1):1-74.

[2] Gahler S. Zur geometric 2-metrische raume. Rev. Roum. Math. Pures et Appl. 1966;11: 664-669.

[3] Gahler S. 2-metrische raume und ihre topologische. Struktur Math. Nachr. 1963;26:115-148.

[4] Ha KS, Cho YJ, White A. Strictly convex and 2-convex 2-normed spaces. Math. Japonica. 1988;33(3):375-384.

[5] Sharma AK. A note on fixed points in 2-metric spaces. Indian J. Pure Appl. Math. 1980;11(2):1580-1583.

[6] Mustafa Z, Sims B. Some remarks concerning D-metric spaces. Proceedings of the International Conferences on Fixed Point Theory and Applications. Valencia (Spain). 2003;189-198.

[7] Mustafa Z, Sims B. A new approach to generalized metric spaces. Journal of Non-linear and Convex Analysis. 2006;7:289-297.

[8] Dhanorkar GA. Applying G-metric space for cantor's intersection and Baire's category theorem. Asian Research Journal of Mathematics. 2017;3(2):1-8.

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