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Certain Geometric Aspects of a Class of Almost Contact Structures on a Smooth Metric Manifold

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Authors' contributions

This work was carried out in collaboration among all the authors. Author OMO designed the study, performed the mathematical analysis and wrote the protocol. Author MB wrote the first draft of the manuscript. Authors OMO, MB and WAW managed the analyses of the study and improvement of the results. All the authors read and approved of final manuscript.

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Abstract

The classification of Smooth Geometrical Manifolds still remains an open problem. The concept of almost contact Riemannian manifolds provides neat descriptions and distinctions between classes of odd and even dimensional manifolds and their geometries. We construct an almost contact structure which is related to almost contact 3-structure carried on a smooth Riemannian manifold (M, g_M) of dimension $(5n + 4)$ such that $\gcd(2, n) = 1$. Starting with the almost contact metric manifolds (N^{4n+3}, g_N) endowed with structure tensors (ϕ_i, ξ_j, η_k) such that $1 \leq i, j, k \leq 3$ of types $(1, 1), (1, 0), (0, 1)$ respectively, we establish that there exists a structure (ϕ_4, ξ_4, η_4) on $(N^{4n+3} \otimes \mathbb{R}^d) \approx M$; $\gcd(4, d) = 1$, $d|2n + 1$, constructed as linear combinations of the three structures on (N^{4n+3}, g_N) . We study some algebraic properties of the tensors of the constructed almost contact structure and further explore the Geometry of the two manifolds $(N^{4n+3} \otimes \mathbb{R}^d) \approx M$ and N^{4n+3} via a !submersion $F : (N^{4n+3} \otimes \mathbb{R}^d) \hookrightarrow (N^{4n+3})$ and the metrics g_M respective g_N

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between them. This provides new forms of Gauss-Weigarten's equations, Gauss-Codazzi equations and the Ricci equations incorporating the submersion other than the First and second Fundamental coefficients only. Fundamentally, this research has revealed that the structure (ϕ_4, ξ_4, η_4) is constructible and it is carried on the hidden compartment of the manifold $M \cong (N^{4n+3} \otimes \mathbb{R}^d)$ $(d|2n+1)$ which is related to the manifold (N^{4n+3}).

Keywords: Almost contact structures; metric manifold.

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1 Introduction

Unless stated otherwise, we shall denote by (M^{5n+4}, g_M) the $5n+4-$ dimensional smooth Riemannian manifold isomorphic to $(N^{4n+3} \otimes \mathbb{R}^d)$ with a compatible metric g_M where the $\gcd(2, n) = 1$, $gcd(4, d) = 1$. This manifold carries 4-almost contact structures. We also denote by (N^{4n+3}, g_N) the $4n+3-$dimensional smooth manifold carrying 3-almost contact structures and compatible with the metric g_N. Other notations are standard and can be found from the references. Due to the epimorphism above, we study the geometry of $(N^{4n+3} \otimes \mathbb{R}^d)$ via the manifold M^{5n+4}.

A $(2n+1)-$dimensional manifold $M \in C^\infty$ is called *contact manifold* if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. The 1-form η is called a contact form of M. It is well known that given a contact form η, there exists a unique vector field ξ satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field $X \in M$ [1]. Chinea and Gonzalez [2] obtained a classification of the $(2n+1)$-dimensional almost contact metric manifold based on $U(n) \times 1$ representation Theory, which is an analogy of the classification of the $2n$-dimensional almost Hermitian manifolds established by Gray and Hervella[3].

Almost 3-contact manifolds were introduced by Kuo[4] and independently, by Udriste [5] . To their class belong also 3-Sasakian ,3-cosymplectic manifolds studied by Boyer and Galicki [6], whose properties were also analyzed by Montano and De Nicola [7]. The almost contact 3-structure has been defined by Kuo, Kuo-Tachibana [4, 8], Tachibana and Yu[9], and studied by them, Yano, Eum and Ki[10], Sasaki [11] among other geometers. Some topics related to almost contact 3-structures have been considered by Ishihara, Konishi [12, 13, 14] and Tanno [15]. It is well known that the product of a manifold with almost contact 3-structure and a straight line admits an almost quaternion structure (cf. [4]). Yano, Ishihara and konishi [16] studied the normality property of almost contact 3-structures in the light the almost quaternion structure (F, G, H).

It has also been shown in [4] that given an almost contact 3-structure (ϕ_i, ξ_i, η_i), $(i = 1, 2, 3)$, $\exists$ a Riemannian metric g compatible with each of them and hence an almost contact metric 3-structures. Moreover, the Reeb vector fields ξ_1, ξ_2, ξ_3 are orthonormal with respect to the compatible metric and the structural group of the tangent bundle is reducible to $Sp(n) \times I_3$. By putting $H = \bigcap_{i=1}^{3} ker(\eta_i)$, we obtain a $4n-$dimensional distribution on M and the tangent bundle splits as the orthogonal sum $TM = H \oplus V$ of horizontal and vertical distribution where $V =< \xi_1, \xi_2, \xi_3 >$.

Blaga [17] has studied almost $k-$contact structure, by pointing out an isoparametric function which can be associated in this framework, by generalizing a similar construction initiated by Mihai and Rosca [18]. From Blag's constructions, an almost $k-$contact manifold is found to be $(n + k + nk)-$dimensional manifold M with k almost contact structures (ϕ_1, η_1, ξ_1) ,..., (ϕ_k, η_k, ξ_k) such that: $\phi_i \circ \phi_j = -\delta_{ij} I_{\Gamma(TM)} + \eta_i \otimes \xi_j + \sum_{l=1}^{k} \epsilon_{ijl} \phi_l$ and $\eta_i(\xi_j) = \delta_{ij}$.Other notions can also be found in [1]. For instance, given an almost contact 3-structure (ϕ_i, ξ_i, ξ_i), define on $M^{2m+1} \times \mathbb{R}$ there are three almost complex structures J_i; $i = 1, 2, 3$ associated to each of the almost contact structures. It is then easy to check that $J_k = J_i J_j = -J_j J_i$. Therefore $M^{2m+1} \times \mathbb{R}$ has an almost quaternionic structure, and hence its dimension is a multiple of 4. Thus the dimension of a manifold with an almost contact 3-structure is of the form $4n + 3$. Tachibana and Yu [9] used this idea to show that there cannot be a fourth almost contact structure (ϕ_4, ξ_4, ξ_4) with $\eta_i(\xi_4) = \eta_4(\xi_i) = 0,\ i = 1, 2, 3$, and satisfying the anticommutativity conditions with the first three structures. To see this, let J_4 be the almost complex structure on $M^{2m+1} \times \mathbb{R}$ constructed using (ϕ_4, ξ_4, ξ_4). Then pairing J_4 with each of J_1, J_2, J_3 yields $J_4J_i = -J_iJ_4,\ i = 1, 2, 3$. This contradicts $J_3J_4 = J_1J_2J_4 = -J_1J_4J_2 = J_4J_1J_2 = J_4J_3$.

In fact, Blaga[17] assumed that the number of almost contact structures carried on a smooth odd dimensional manifold will always be odd so that formular $Dim(M) = n+nk+k$ holds for a $k-$almost contact manifold. This may not necessarily be the general case since the result below also follows:

Theorem 1.1. *The dimension of a manifold with an almost contact $k-$structure is of the form $n + (n - 1)k + 2k$ for an even k.*

This research therefore demonstrates that it is possible to construct a fourth almost contact structure (ϕ_4, ξ_4, η_4) in terms of the first three structures iff it is carried on a manifold related to N^{4n+3} and given by $M^{5n+4} \cong N^{4n+3} \otimes \mathbb{R}^d :\ \gcd(2, n) = 1,\ \gcd(4, d) = 1$.

2 Fundamental Results

These preliminaries are standard and can be found in the references eg [1]:

Let M be a $(2n + 1)$-dimensional differentiable manifold and ϕ, ξ, η be a field of endomorphisms of the tangent spaces TM as a $(1, 1)-$tensor field, a vector field and a $1-$form on M respectively. If a triple (ϕ, ξ, η) satisfies the two conditions

$$\eta(\xi) = 1 \tag{2.1}$$

$$\phi^2(X) = -X + \eta(X)\xi \tag{2.2}$$

for any vector field X on M, (ϕ, ξ, η) is called *an almost contact structure* and M is called *an almost contact manifold*.

Note that every almost contact manifold must have a non-singular vector ξ over M by the definition.

Proposition 2.1. *For an almost contact structure (ϕ, ξ, η) on M,*

$$\phi(\xi) = 0.............(i), \eta \circ \xi = 0.......(ii), rank(\phi) = 2n........(iii) \tag{2.3}$$

Proof. For a non-singular vector field ξ,

$$\phi^2(\xi) = -\xi + \eta(\xi)\xi = -\xi + 1.\xi = 0 \tag{2.4}$$

and

$$0 = \phi^2\phi(\xi) = -\phi(\xi) + \eta(\phi(\xi))\xi \tag{2.5}$$

So we have

$$\phi(\xi) = \eta(\phi(\xi))\xi \tag{2.6}$$

From 2.4, it follows that $\phi(\xi) = 0$ or $\phi(\xi)$ is a non-zero vector field whose image is zero. Suppose $\phi(\xi)$ is a nonzero vector field which goes to 0. In this case $\eta(\phi(\xi))$ is not zero. If $\eta(\phi(\xi)) = 0$, then $\phi(\xi) = 0$ in 2.6, which is a contradiction to the assumption. Then, by 2.6,

$$\phi^2(\xi) = \phi(\phi(\xi)) = \phi(\eta(\phi(\xi))\xi) = \eta(\phi(\xi)).\phi(\xi) = \eta(\phi(\xi)).\eta(\phi(\xi)).\xi = \{\eta(\phi(\xi))\}^2.\xi$$

and we have a nontrivial $\phi^2(\xi)$ because $\eta(\phi(\xi))$ and ξ are non-zero. But this contradicts to the fact that $\phi^2(\xi) = 0$. Therefore we conclude that $\phi(\xi) = 0$ and (i) is proved.

Next, from 2.2, we get,

$$\phi^3(X) = \phi(\phi^2(X)) = \phi(-X + \eta(X)\xi) = \phi(-X) + \phi(\eta(X)\xi) = -\phi(X) + \phi(\eta(X)\xi)$$

for any vector X. On the other hand, we rewrite $\phi^3(X)$ as;

$$\phi^3(X) = \phi^2(\phi(X)) = -\phi(X) + \eta(\phi(X))\xi$$

$$\Rightarrow \eta(\phi(X))\xi = \phi^3(X) + \phi(X) = -\phi(X) + \eta(\phi(X))\xi + \phi(X) = \eta(\phi(X))\xi = 0$$

from the previous result $\phi(\xi) = 0$. Therefore $\eta \circ \phi = 0$ for any vector X.

We now claim that $rank(\phi) = 2n$. Since $\phi(\xi) = 0$, it is clear that ϕ has dimension less than or equal to $2n$. Suppose there exists another vector $\mathbf{X}$ of M such that $\phi(\mathbf{X}) = 0$. Then $\phi^2(\mathbf{X}) = \phi\underbrace{(\phi(\mathbf{X}))}_{0} = -\mathbf{X} + \eta(\mathbf{X})R$ implies that $\mathbf{X} = \eta(\mathbf{X})\xi$ □

We next consider a metric on a manifold with an almost contact structure. We know that if M is paracompact then M admits a Riemannian metric tensor and denote it by $h^{'}$. We obtain a Riemannian metric h by setting

$$h(X,Y) = h^{'}(\phi^2(X), \phi^2(Y)) + \eta(X)\eta(Y) = h^{'}[-X + \eta(X)\xi, -Y + \eta(Y)\xi] + \eta(X)\eta(Y)$$

and we have the following:

Lemma 2.1. *Every almost contact manifold M admits a Riemannian metric tensor h such that*

$$h(X, \xi) = \eta(X) \tag{2.7}$$

for every vector field X on M

Proof. Let $Y = \xi$. Then, by definition of h,

$$h(X,\xi) = h^{'}(\phi^2(X), \underbrace{\phi^2(\xi)}_{0}) + \eta(X)\underbrace{\eta(\xi)}_{1} = \eta(X)$$

We also have, $h(\xi, Y) = \eta(Y)$ by setting $X = \xi$ and $h(\xi,\xi) = \eta(\xi) = 1$ as required. □

Proposition 2.2. *Every almost contact manifold M admits a Riemannian metric tensor field g such that*

$$g(\phi(X), \phi(Y)) = g(X,Y) - \eta(X)\eta(Y) \tag{2.8}$$

Proof. Define g by $g(X,Y) = \frac{1}{2}(h(X,Y) + h(\phi X, \phi Y) + \eta(X)\eta(Y))$ with the same Riemannian metric h as $h(X,\xi) = \eta(X)$. We rewrite $g(\phi(X), \phi(Y))$ as:

$$g(\phi X, \phi Y) = \frac{1}{2}(h(\phi X, \phi Y) + h(\phi^2 X, \phi^2 Y) + \eta(\phi X)\eta(\phi Y)).$$

Since $\eta \circ \phi = 0$,

$$g(\phi X, \phi Y) = \frac{1}{2}(h(\phi X, \phi Y) + h(-X + \eta(X)\xi, -Y + \eta(Y)\xi))$$

$$= \frac{1}{2}(h(\phi X, \phi Y) + h(X,Y) - \eta(Y)\underbrace{h(X,\xi)}_{\eta(X)} - \eta(X)(\underbrace{h(\xi,Y)}_{\eta(Y)}) + \eta(X)\eta(Y)\underbrace{h(\xi,\xi)}_{1})$$

$$= \frac{1}{2}(h(\phi X, \phi Y) + h(X,Y) - \eta(Y)\eta(X) - \eta(X)\eta(Y) + \eta(X)\eta(Y))$$

$$= \frac{1}{2}(h(\phi X, \phi Y) + h(X,Y) - \eta(Y)\eta(X))$$
$$= g(X,Y) - \eta(X)\eta(Y)$$

□

Remark 2.1. Since $\eta \circ \phi = 0$,

$$g(\phi X, Y) = g(\phi^2 X, \phi Y) + \eta(\phi(X))\eta(Y)$$
$$= g(\phi^2 X, \phi Y)$$
$$= g(-X + \eta(X)\xi, \phi Y)$$
$$= g(-X, \phi Y) + \eta(X) g(\xi, \phi Y)$$
$$= -g(X, \phi Y)$$

because $g(\xi, \phi Y) = g(\underbrace{\phi\xi}_{0}, \phi^2 Y) + \eta(\xi)\underbrace{\eta(\phi Y)}_{0} = 0$. Hence, ϕ is a skew-symmetric tensor field with respect to the metric g. That is ,

$$g(\phi X, Y) + g(X, \phi Y) = 0.$$

If M admits a tensor field (ϕ, ξ, η, g) shown in the previous previous proposition, then we say that M has an almost contact metric structure (ϕ, ξ, η, g) and is called an almost contact metric manifold.

Proposition 2.3. *A $(2n+1)-$dimensional manifold M admits an almost contact structure (ϕ, ξ, η) if and only if the structure group of its tangent bundle reduces to $U(n) \times 1$.*

Proof. Let ξ be a non-singular vector field on the almost contact manifold M and $V = \{v_1, ..., v_n, \phi v_1, ..., \phi v_n, \xi\}$ be an orthonormal basis of M. Then we have a matrix g as follows:

$$\begin{pmatrix} < v_1, v_2 > & \cdots & < v_1, v_n > & < v_1, \phi v_1 > & \cdots & < v_1, \phi v_n > & < v_1, \xi > \\ < v_2, v_1 > & \cdots & < v_2, v_n > & < v_2, \phi v_1 > & \cdots & < v_2, \phi v_n > & < v_1, \xi > \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ < v_n, v_1 > & \cdots & < v_n, v_n > & < v_n, \phi v_1 > & \cdots & < v_n, \phi v_n > & < v_n, \xi > \\ < \phi v_1, v_1 > & \cdots & < \phi v_n, v_n > & < \phi v_1, \phi v_1 > & \cdots & < \phi v_1, \phi v_n > & < \phi v_1, \xi > \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ < \phi v_n, v_1 > & \cdots & < \phi v_n, v_n > & < \phi v_n, \phi v_1 > & \cdots & < \phi v_n, \phi v_n > & < \phi v_n, \xi > \\ < \xi, v_1 > & \cdots & < \xi, v_n > & < \xi, \phi v_1 > & \cdots & < \xi, \phi v_n > & < \xi, \xi > \end{pmatrix}$$

Since $g_{ij} =< v_i, v_j >=< \phi v_i, \phi v_j >= \delta_{ij}$ and $g_{ij} =< \phi v_i, v_j >= 0$ for all i, j, the matrix g is of the form $g = \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and we see that $\phi = \begin{pmatrix} 0 & I_n & 0 \\ -I_n 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ because the rank of $\phi = 2n$. Moreover,

$$\phi(V) = \begin{pmatrix} 0 & I_n & 0 \\ -I_n 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} (v_1, ..., v_n, \phi v_1, ..., \phi v_n, \xi) = (\phi v_1, ..., \phi v_n, -v_1, ..., -v_n, 0)$$

and

$$\phi(V) = \phi(\phi v_1, ..., \phi v_n, -v_1, ..., -v_n, 0) \Rightarrow \phi(\phi(v_i)) = -v_i, \phi^2(\xi) = 0$$

Now, we take another orthonormal basis $\{v_1^{'}, ..., v_n^{'}, \phi v_1^{'}, ..., \phi v_n^{'}, \xi\}$ of M with the same g and ϕ and put

$$r v_1 = v_1^{'}, ..., r v_n = v_n^{'}, r\phi v_1 = \phi v_1^{'}, ..., r\phi v_n = \phi v_n^{'}, r\xi = \xi$$

We claim that the matrix $r : M \to M$ must have the form $r = \begin{pmatrix} A_n & B_n & 0 \\ -B_n & A_n & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Let $r : M \to M$ be of the form $r = \begin{pmatrix} A_n & B_n & 0 \\ C_n & D_n & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then, for a basis V,

$$r(V) = \begin{pmatrix} A_n & B_n & 0 \\ C_n & D_n & 0 \\ 0 & 0 & 1 \end{pmatrix} (v_1, ..., v_n, \phi v_1, ..., \phi v_n, \xi) = (v_1^{'}, ..., v_n^{'}, \phi v_1^{'}, ..., \phi v_n^{'}, \xi)$$

Substituting X for $n-$coordinates $v_1...v_n$ and Y for another $n-$coordinates $e_1^{'}, ..., e_n^{'}$ give us a system of equations as follows of the form $A_n(X) + B_n(\phi(X)) = Y$ and $C_n(X) + D_n(\phi(X)) = \phi(Y)$. Solving for C_n and D_n gives $C_n = B_n$ and $D_n = A_n$. Therefore, the structure group of the tangent bundle of M can be reduced to $U_n \times 1$.

Conversely, if the structure group of the tangent bundle of M can be reduced to $U_n \times 1$, then we can define $g = \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\phi = \begin{pmatrix} 0 & I_n & 0 \\ -I_n 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We can also give a vector field ξ by $(\underbrace{0, 0, ..., 0}_{2n}, 1)$ and a $1-$form η by an associated $1-$form of a vector field ξ. They satisfy the desired conditions. □

Corollary 2.2. *The previous result holds necessarily and the structure group of the tangent bundle of M reduces to $U(n) \times 1$ and every element of $U(n) \times 1$ has positive determinant.*

3 The Fourth Structure (ϕ_4, ξ_4, η_4) on $M^{5n+4} \cong N^{4n+3} \otimes \mathbb{R}^d$

The following results are important in the sequel:

Proposition 3.1. *Let $\phi_1, \phi_2 \in T_{(1,1)}$, $\xi_1, \xi_2 \in TM$ and $\eta_1, \eta_2 \in TM^*$. Suppose (ϕ_1, ξ_1, η_1) and (ϕ_2, ξ_2, η_2) are both almost contact structures and satisfy:*

$$\phi_1\phi_2 + \phi_2\phi_1 = \eta_1 \otimes \xi_2 + \eta_2 \otimes \xi_1, \ \phi_1\xi_2 + \phi_2\xi_1 = 0,$$

$$\eta_1 \circ \phi_2 + \eta_2 \circ \phi_1 = 0, \ \eta_1(\xi_2) = 0, \ \eta_2(\xi_1) = 0$$

then the sets (ϕ_1, ξ_1, η_1) and (ϕ_2, ξ_2, η_2) are said to define an almost contact 3-structure.

Proof. Putting $\phi_3 = \phi_1\phi_2 - \eta_2 \otimes \xi_1 = -\phi_2\phi_1 + \eta_1 \otimes \xi_2$, $\xi_3 = \phi_1\xi_2 = -\phi_2\xi_1$ and $\eta_3 = \eta_1 \circ \phi_2 = \eta_2 \circ \phi_1$. We can easily verify that (ϕ_3, ξ_3, η_3) defines an almost contact structure as follows:

Let $X \in TM$ and $\phi_3^2(X) = -I + \eta(X)\xi$, then we have

$$\eta_3(\xi_3) = -\eta_2 \circ \phi_1(\xi_3) = -\eta_2(\phi_1(\xi_3)) = -\eta_2(\phi_1\xi_3)$$

But

$$\phi_1\eta_3 = \phi_1(\phi_1\xi_2) = \phi_1^2\xi_2 = -\xi_2 + \eta_1(\xi_2)\xi_1 = -\xi_2 + 0 = -\xi_2$$

So

$$-\eta_2(\phi_1\xi_3) = -\eta_2(-\xi_2) = \eta_2(\xi_2) = 1, \ \Rightarrow \eta_3(\xi_3) = 1$$

Next,

$$\phi_3\xi_3 = \phi_3(-\phi_2\xi_1) = \phi_3(-\phi_2(X)\xi_1) = (\phi_1\phi_2(X)) - \underbrace{\eta_2(X)\xi_1}_{0}(-\phi_2(X)\xi_1)$$

$$= -\phi_1(\phi_2^2(X)\xi_1 - 0) = -\phi_1(\phi_2^2\xi_1) = -\phi_1(-\xi_1 + \underbrace{\eta_2(\xi_1)}_{0}\xi_2) = \phi_1\xi_1 = 0$$

$$\Rightarrow \phi_3\xi_3 = 0$$

Next,

$$\eta_3 \circ \phi_3(X) = \eta_3(\phi_3(X)) = \eta_3(\phi_1\phi_2(X) - \eta_2(X)\xi_1) = \eta_3((\phi_1\phi_2(X)) - \eta_2(X)\eta_3(\xi_1))$$

$$= -\eta_2 \circ \phi_1(\phi_1(\phi_2 X)) + \eta_2(X)\eta_2 \circ \phi_1(\xi_1) - \underbrace{\eta_2(\phi_1^2\phi_2 X) + \eta_2(X)\eta_2\underbrace{(\phi_1(\xi_1))}_{0}}_{0}$$

$$= -\eta_2(-\phi_2 X + \eta_1(\phi_2 X)\xi_1) = \eta_2(\phi_2 X) - \eta_1(\phi_2 X)\underbrace{\eta_2(\xi_1)}_{0} = \eta_2(\phi_2(X)) = 0$$

$$\Rightarrow \eta_3 \circ \phi_3(X) = 0.$$

Furthermore, we can see that

$$\phi_1 = \phi_2\phi_3 - \eta_3 \otimes \xi_2 = -\phi_3\phi_2 + \eta_2 \otimes \xi_3,\ \phi_2 = \phi_3\phi_1 - \eta_1 \otimes \xi_3 = -\phi_1\phi_3 + \eta_3 \otimes \xi_1$$

$$\xi_1 = \phi_2\xi_3 = -\phi_3\xi_2,\ \xi_2 = \phi_3\xi_1 = -\phi_1\xi_3$$

$$\eta_1 = \eta_2 \circ \phi_3 = -\eta_3 \circ \phi_2,\ \eta_2 = \eta_3 \circ \phi_1 = -\eta_1 \circ \phi_3$$

Therefore, any two of (ϕ_1, ξ_1, η_1), (ϕ_2, ξ_2, η_2) and (ϕ_3, ξ_3, η_3) define essentially the same almost contact 3-structure. In this sense, we say that such almost contact structures (ϕ_i, ξ_i, η_i), $(i = 1, 2, 3)$ define in M an almost contact 3-structure. □

Theorem 3.1. *(cf.[4]) If a differentiable manifold admits 2 almost contact structures* (ϕ_i, ξ_i, η_i) : $i = 1, 2$, *satisfying:* $\eta_1(\xi_2) = \eta_2(\xi_1) = 0$, $\phi_1\xi_2 = -\phi_2\xi_1 = \xi_3$, $\eta_1 \circ \phi_2 = -\eta_2 \circ \phi_1 = \eta_3$ *and* $\phi_1\phi_2 - \eta_2 \otimes \xi_1 = -\phi_2\phi_1 + \eta_1 \otimes \xi_2 = \phi_3$ *then it admits a third almost contact structure* (ϕ_3, ξ_3, η_3).

3.1 The construction of (ϕ_4, ξ_4, η_4)

Following the results of Tachibana and Yu [9], in this subsection, starting with 3-almost contact structures, we construct an almost contact structure (ϕ_4, ξ_4, η_4) such that $\eta_i(\xi_4) \neq \eta_4(\xi_i) \neq 0$, $i = 1, 2, 3$, necessarily. The dimension of the manifold carrying the 4-almostcontact structures (ϕ_1, ξ_1, η_1), (ϕ_2, ξ_2, η_2), (ϕ_3, ξ_3, η_3), (ϕ_4, ξ_4, η_4) must be of the form $5n + 4$.
The following results are useful in our construction:

Proposition 3.2. *(cf.[1]) About each point of an almost contact manifold* M^d, *there exists local coordinates* $(x_1, ..., x_n, y_1, ..., y_n, f)$ *with respect to which* $\eta = df - \sum_{i=1}^n y_i dx_i$

Proof. In some coordinate neighborhood choose an open-ball transverse to ξ such that $d\eta$ is symplectic on this ball, and hence there exist local coordinates $(x_1, ..., x_n, y_1, ..., y_n, f)$ such that $d\eta = \sum dx_i \wedge dy_i$. Now $d(\eta + \sum_{i=1}^n y_i dx_i) = 0$ so that $\eta + \sum_{i=1}^n y_i dx_i = df$ for some function f. Clearly, $\eta \wedge (d\eta)^n = df \wedge dx_1 \wedge ... \wedge dx_n \wedge dy_1 \wedge ... \wedge dy_n \neq 0$. Therefore df is independent of $dx_1 dy_i$ and hence we can regard x_i, y_i and f as a coordinate system. □

Proposition 3.3. *(***Existence Result***) Let* $(M^{5n+4}, \phi_i, \xi_i, \eta_i, g)$; $i = 1, 2, 3$ *be an almost contact metric 3-structure. On* $M^{5n+3} \times \mathbb{R}$, *when* $2 \mid m$, *we define an almost complex structure* J_i *by*

$$J_1\big(X, f\frac{d}{dt}\big) = \big(\phi_1 X - f\xi_1, \eta_1(X)\frac{d}{dt}\big),\ J_2\big(X, f\frac{d}{dt}\big) = \big(\phi_2 X - f\xi_2, \eta_2(X)\frac{d}{dt}\big) \tag{3.1}$$

$$J_3\big(X, f\frac{d}{dt}\big) = \big(\phi_3 X - f\xi_3, \eta_3(X)\frac{d}{dt}\big)$$

where $X \in \Gamma(TM)$ and $f \in C^\infty(M^{5n+3} \times \mathbb{R})$. Let $J_i;\ i = 1, .., , 3$ be integrable, that is $[J_i, J_i] = 0$ so that (ϕ_i, ξ_i, η_i) is hypernormal. Suppose there exist another almost complex structure J_4 such that $J_4\big(X, f\frac{d}{dt}\big) = \big(\phi_4 X - f\xi_4, \eta_4(X)\frac{d}{dt}\big)$ and $[J_i, J_i] = 0$, then (ϕ_4, ξ_4, η_4) is an almost contact structure. Moreover if $J_3J_4 = J_1J_2J_4 = -J_1J_4J_2 = J_4J_1J_2 = J_4J_3$, then (ϕ_4, ξ_4, η_4) defines an almost contact structure whose field of endomorphism satisfies the anticommutativity condition with the other three.

We now proceed with our construction as follows:

Let $(\phi_1, \xi_1, \eta_1),\ (\phi_2, \xi_2, \eta_2,\ \phi_3, \xi_3, \eta_3)$ be almost contact 3-structures on M^{5n+4}. From Theorem 3.1, we see that

$$\phi_1\phi_2 + \phi_2\phi_1 = \eta_1 \otimes \xi_2 + \eta_2 \otimes \xi_1 = 0\ \phi_1\xi_2 + \phi_2\xi_1 = 0 \tag{3.2}$$

so that $\phi_1 = \phi_2\phi_3 - \eta_3 \otimes \xi_2 = -\phi_3\phi_2 + \eta_2 \otimes \xi_3,\ \phi_2 = \phi_3\phi_1 - \eta_1 \otimes \xi_3 = -\phi_1\phi_3 + \eta_3 \otimes \xi_1$ and $\phi_3 = \phi_1\phi_2 - \eta_2 \otimes \xi_1 = -\phi_2\phi_1 + \eta_1 \otimes \xi_2$. Similar descriptions can be given for ξ_i and η_i according to the same result. We need to construct (ϕ_4, ξ_4, η_4) such that each of the respective tensors is expressed in terms of the first three above.

With obvious identifications, we see that $\exists$ some endomorphism constructible from $\phi_1,\ \phi_2,\ \phi_3$ which are pairwise anti-commutative and thus:

$$\begin{aligned}&\phi_1\phi_2 + \phi_2\phi_1 + \phi_1\phi_3 + \phi_3\phi_1 + \phi_2\phi_3 + \phi_3\phi_2 = \eta_1 \otimes \xi_2 + \eta_2 \otimes \xi_1 + \\ &\eta_1 \otimes \xi_3 + \eta_3 \otimes \xi_1 + \eta_2 \otimes \xi_3 + \eta_3 \otimes \xi_2 = 0\end{aligned} \tag{3.3}$$

Exhausting the permutations of all the possible combinations of 3.3, results to possible constructions for ϕ_4, as follows:

$$\begin{aligned}\phi_4 &= \phi_1\phi_2 + \phi_2\phi_3 + \phi_3\phi_1 - \big(\eta_2 \otimes \xi_1 + \eta_3 \otimes \xi_2 + \eta_1 \otimes \xi_3\big) \\ &= -\big(\phi_2\phi_1 + \phi_3\phi_2 + \phi_1\phi_3\big) + \eta_1 \otimes \xi_2 + \eta_2 \otimes \xi_3 + \eta_3 \otimes \xi_1\end{aligned} \tag{3.4}$$

Similarly,

$$\xi_4 = \phi_1\xi_2 + \phi_2\xi_3 + \phi_3\xi_1 = -\big(\phi_2\xi_1 + \phi_3\xi_2 + \phi_1\xi_3\big) \tag{3.5}$$

But

$$\eta_1 \circ \phi_2 + \eta_2 \circ \phi_1 + \eta_1 \circ \phi_3 + \eta_3 \circ \phi_1 + \eta_2 \circ \phi_3 + \eta_3 \circ \phi_2 = 0$$

and $\eta_i(\xi_j) = \eta_j(\xi_i) = 0;\ i \neq j,\ \eta_i(\xi_i) = 1,\ \eta_i(\phi_i) = 0\ \forall i = 1, 2, 3$ so we need an appropriate η_4 from the construction such that the aggregate (ϕ_4, ξ_4, η_4) is an almost contact structure. By inspection, we immediately see that

$$\eta_4 = \frac{1}{3}\big(\eta_1 \circ \phi_2 + \eta_2 \circ \phi_3 + \eta_3 \circ \phi_1\big) = -\frac{1}{3}\big(\eta_2 \circ \phi_1 + \eta_3 \circ \phi_2 + \eta_1 \circ \phi_3\big) \tag{3.6}$$

Proposition 3.4. *Let n be an odd integer. The aggregate (ϕ_4, ξ_4, η_4), given by the construction above is the unique fourth almost contact structure on M^{5n+4} such that $\eta_i(\xi_4) = \eta_4(\xi_i);\ i = 1, 2, 3$.*

Proof. Recall that $\xi_1 = \phi_2\xi_3 - \phi_3\xi_2,\ \xi_2 = \phi_3\xi_1 - \phi_1\xi_3,\ \xi_3 = \phi_1\xi_2 - \phi_2\xi_1$. Let $\phi_4^2 = -I + \eta_4 \otimes \xi_4$. We need to show that $\eta_4(\xi_4) = 1,\ \phi_4\xi_4 = 0$ and $\eta_4 \circ \phi_4 = 0$. Clearly,

$$\begin{aligned}&\eta_4(\xi_4) = \frac{1}{3}\big(\eta_1 \circ \phi_2 + \eta_2 \circ \phi_3 + \eta_3 \circ \phi_1\big)\big(\phi_1\xi_2 + \phi_2\xi_3 + \phi_3\xi_1\big) \\ &= \frac{1}{3}\big(\{\eta_1(\phi_2\xi_1) + \eta_1(\phi_2\xi_2) + \eta_1(\phi_2\xi_3)\} + \{\eta_2(\phi_3\xi_1) + \eta_2(\phi_3\xi_2) + \eta_2(\phi_3\xi_3)\} + \\ &\{\eta_3(\phi_1\xi_1) + \eta_3(\phi_1\xi_2) + \eta_3(\phi_1\xi_3)\}\big) \\ &= \frac{1}{3}\big(-\eta_1\xi_3 + \eta_1\xi_1 + \eta_2\xi_2 - \eta_2\xi_1 + \eta_3\xi_3 - \eta_3\xi_2\big) = \frac{1}{3}(3) = 1\end{aligned} \tag{3.7}$$

Next,

$$\begin{aligned}
\phi_4\xi_4 &= \big(\phi_1\phi_2 + \phi_2\phi_3 + \phi_3\phi_1 - \big(\eta_2 \otimes \xi_1 + \eta_3 \otimes \xi_2 + \eta_1 \otimes \xi_3\big)\big)\big(\xi_1 + \xi_2 + \xi_3\big) \\
&= \big(\phi_1\phi_2\xi_1 + \phi_1\phi_2\xi_2 + \phi_1\phi_2\xi_3 + \phi_2\phi_3\xi_1 + \phi_2\phi_3\xi_2 + \phi_2\phi_3\xi_3 + \phi_3\phi_1\xi_1 + \phi_3\phi_1\xi_2 \\
&+ \phi_3\phi_1\xi_3\big) - \big(\eta_2 \sum_{i=1}^{3}(\xi_i) \otimes \xi_1 + \eta_3 \sum_{i=1}^{3}(\xi_i) \otimes \xi_2 + \eta_1 \sum_{i=1}^{3}(\xi_i) \otimes \xi_3\big) \\
&= \big(- \phi_1\xi_3 - \phi_2\xi_1 - \phi_3\xi_2\big) - \big(\sum_{i=1}^{3}(\xi_i)\big) = \big(\sum_{i=1}^{3}(\xi_i)\big) - \big(\sum_{i=1}^{3}(\xi_i)\big) = 0
\end{aligned} \tag{3.8}$$

Finally,

$$\begin{aligned}
\eta_4 \circ \phi_4 &= \frac{1}{3}\big(\eta_1 \circ \phi_2 + \eta_2 \circ \phi_3 + \eta_3 \circ \phi_1\big)\big(\phi_4\big) = \frac{1}{3}\big(\big(\eta_1 \circ \phi_2 + \eta_2 \circ \phi_3 + \eta_3 \circ \phi_1\big)\big(\phi_4\big)\big) \\
&= \frac{1}{3}\big\{\big(\eta_1 \circ \phi_2 + \eta_2 \circ \phi_3 + \eta_3 \circ \phi_1\big)\big(\phi_1\phi_2 + \phi_2\phi_3 + \phi_3\phi_1\big) - \\
&\big(\eta_1 \circ \phi_2 + \eta_2 \circ \phi_3 + \eta_3 \circ \phi_1\big)\big(\eta_2 \otimes \xi_1 + \eta_3 \otimes \xi_2 + \eta_1 \otimes \xi_3\big)\big\} \\
&= \frac{1}{3}\big\{\eta_1(\phi_2\phi_2\phi_3) + \eta_2(\phi_3\phi_3\phi_1) + \eta_1(\phi_1\phi_1\phi_2)\big\} - \frac{1}{3}\big\{\eta_1\phi_2(\eta_2 \otimes \xi_1) + \eta_1\phi_2(\eta_3 \otimes \xi_2) + \\
&\eta_1\phi_2(\eta_1 \otimes \xi_3) + \eta_2\phi_3(\eta_2 \otimes \xi_1) + \eta_2\phi_3(\eta_3 \otimes \xi_2) + \eta_2\phi_3(\eta_1 \otimes \xi_3) + \eta_3\phi_1(\eta_2 \otimes \xi_1) + \\
&\eta_3\phi_1(\eta_3 \otimes \xi_2) + \eta_3\phi_1(\eta_1 \otimes \xi_3)\big\}
\end{aligned} \tag{3.9}$$

Applying a vector field $\xi_i \in \{\xi_1, \xi_2, \xi_3\}$ to equation 3.9, consider ξ_2 say, we have:

$$\begin{aligned}
&\frac{1}{3}\big\{\eta_1(\phi_2\phi_2\phi_3\xi_2) + \eta_2(\phi_3\phi_3\phi_1\xi_2) + \eta_1(\phi_1\phi_1\phi_2\xi_2)\big\} - \frac{1}{3}\big\{\eta_1\phi_2(\eta_2(\xi_2) \otimes \xi_1) + \eta_1\phi_2(\eta_3(\xi_2) \otimes \xi_2) \\
&+ \eta_1\phi_2(\eta_1(\xi_2) \otimes \xi_3) + \eta_2\phi_3(\eta_2(\xi_2) \otimes \xi_1) + \eta_2\phi_3(\eta_3(\xi_2) \otimes \xi_2) + \eta_2\phi_3(\eta_1(\xi_2) \otimes \xi_3) \\
&+ \eta_3\phi_1(\eta_2(\xi_2) \otimes \xi_1) + \eta_3\phi_1(\eta_3(\xi_2) \otimes \xi_2) + \eta_3\phi_1(\eta_1(\xi_2) \otimes \xi_3)\big\} \\
&= \frac{1}{3}\big(\eta_1\phi_2(-\phi_2\xi_1)\big) - \frac{1}{3}\big(\eta_1(\phi_2\xi_1) + \eta_2(\phi_3\xi_1) + \eta_3(\phi_1\xi_1)\big) \\
&= \frac{1}{3}\big(\eta_1(\phi_2\xi_3)\big) - \frac{1}{3}\big(- \eta_1\xi_3 + \eta_2\xi_2)\big) = \frac{1}{3}\big(\eta_1\xi_1 - \eta_2\xi_2\big) = 0
\end{aligned} \tag{3.10}$$

Thus (ϕ_4, ξ_4, η_4) is an almost contact structure on M^{5n+4} as required □

Corollary 3.2. *Let $(M^{5n+4}, g_M) \cong (N^{4n+3} \otimes \mathbb{R}^d, g_M)$ be the metric manifold discussed in this paper, containing almost contact three structures $(\phi_i, \xi_i, \eta_i); i = 1, 2, 3$ where ϕ_i are the 3 $(1, 1)$ tensors, ξ_i the 3 vector fields and η_i the three 1−forms respectively whose constructions are discussed in section 3. For an odd integer n, (M^{5n+4}, g) contains an almost contact structure (ϕ_4, ξ_4, η_4) constructible from (ϕ_i, ξ_i, η_i); $i = 1, 2, 3$ whose tensors are given by:*

$$\phi_4 = \sum_{i=1,2,3,\ j=2,3,1} \big(\phi_i\phi_j\big) - \sum_{i=1,2,3,\ j=2,3,1} \big(\eta_j \otimes \xi_i\big) = \sum_{i=1,2,3,\ j=2,3,1} -\big(\phi_j\phi_i\big) + \sum_{i=1,2,3,\ j=2,3,1} \big(\eta_i \otimes \xi_j\big)$$

$$\xi_4 = \sum_{i=1,2,3,\ j=2,3,1} \big(\phi_i\xi_j\big) = \sum_{i=1,2,3,\ j=2,3,1} -(\phi_j\xi_i)$$

$$\eta_4 = \frac{1}{3}\big(\sum_{i=1,2,3,\ j=2,3,1} (\eta_i \circ \phi_j)\big) = \frac{1}{3}\big(\sum_{i=1,2,3,\ j=2,3,1} -(\eta_j\phi_i)\big)$$

Moreover, $\eta_i(\xi_4) = \eta_4(\xi_i) = 1,\ \forall i = 1, 2, 3.$

3.2 The Associated Metric g_M of Tangent Bundle $T(M^{5n+4})$

Proposition 3.5. *Let g^I, g^{II}, g^{III}, g^{IV} be the positive definite metrics associated to the structures (ϕ_1, ξ_1, η_1), ..., (ϕ_4, ξ_4, η_4) respectively in the differentiable manifold M of almost contact 4-structure. Then there exists an associated metric of the structure such that if $X, Y \in TM$ then $\forall\ i = 1, 2, 3, 4$,*

$$g(X,Y) = \frac{1}{5}\big\{g^{IV}(X,Y) + \sum_{i=1}^{4}\big\{g^{IV}(\phi_i(X), \phi_i(Y)) + \eta_i(X) + \eta_i(Y)\big\}\big\} \tag{3.11}$$

Proof. Let g^I be the associated metric t (ϕ_1, ξ_1, η_1) then is easy to see that g^{II}, g^{III}, g^{IV} can be defined as:

$$g^{II}(X,Y) = g^I\big(X - \eta_2(X)\xi_2, Y - \eta_2(Y)\xi_2\big) + \eta_2(X)\eta_2(Y)$$

$$g^{III}(X,Y) = g^{II}\big(X - \eta_3(X)\xi_3, Y - \eta_3(Y)\xi_3\big) + \eta_3(X)\eta_3(Y)$$

and

$$g^{IV}(X,Y) = g^{III}\big(X - \eta_4(X)\xi_4, Y - \eta_4(Y)\xi_4\big) + \eta_4(X)\eta_4(Y)$$

so that

$$5g(X,Y) = g^{IV}(X,Y) + \sum_{i=1}^{4}\big\{g^{IV}(\phi_i(X), \phi_i(Y)) + \eta_i(X) + \eta_i(Y)\big\}$$

□

4 Geometric Relationships between (M^{5m+4}, g_M) and (N^{4n+3}, g_N) via Submersion

In this section, accordingly, we denote by g_M the metric compatible with $M^{5m+4} \cong N^{4n+3} \otimes \mathbb{R}^d$ defined by:

$$g_M(X,Y) = \frac{1}{5}\big\{g^{IV}(X,Y) + \sum_{i=1}^{4}\big\{g^{IV}(\phi_i(X), \phi_i(Y)) + \eta_i(X) + \eta_i(Y)\big\}\big\}$$

and by g_N the metric compatible with N^{4n+3} defined by

$$g_N(X,Y) = g^{III}\big(X - \eta_4(X)\xi_4, Y - \eta_4(Y)\xi_4\big) + \eta_4(X)\eta_4(Y)$$

Submersions between these Riemannian manifolds are useful for comparing geometric structures between them.

Foundationally, a smooth map $F : (M, g_M) \to (N, g_N)$ between the Riemannian manifolds (M, g_M) and (N, g_N) is called isometric immersion (submanifold) if F_* is injective and

$$g_N(F_*X, F*Y) = g_M(X,Y) \tag{4.1}$$

for $X, Y \in TM$ and F_* a derivative map.

A smooth map $F : (M, g_M) \to (N, g_N)$ is called a Riemannian submersion if F_* is onto and satisfies equation 4.1, for vector fields tangent to the horizontal space $(kerF_*)^\perp$.

Let $F : (M, g_M) \to (N, g_N)$ be a smooth map between the above Riemannian manifolds M, N such that $0 < rankF < min(5n+4, 4n+3)$, for odd n, where the dimension of $M = 5n+4$ and dimension of $N = 4n+3$, then we denote by $kerF_*$ the kernel space of F^* and consider the orthogonal complementary space $\mathcal{H} = (kerF_*)^\perp$ to $kerF_*$. Then, the tangent bundle of M has the following decomposition:

$$TM = KerF_* \oplus \mathcal{H}$$

Similarly, we consider the orthogonal complementary space $(rangeF_*)^\perp$ to range F_* in the tangent bundle TN. Since, $rankF < min(5n+4, 4n+3)$, we always have that $(rangeF_*)^\perp \neq 0$. Thus TN has the following decomposition:

$$TN = (rangeF_*) \oplus (rangeF_*)^\perp.$$

There is a set of equations that can be used to describe the relationships between invariant quantities on the empirical submanifolds N and ambient manifold M when the Riemannian connection is used. These relationships can expressed by the Gauss' formulae, Weingartens' formulae and the equations of Gauss, Codazzi and Ricci. The said equations can be thus extended to submersion between M^{5n+4} and N^{4n+3}.

To do this, we recall the pullback connection along a map and find the second fundamental form of the map which is used to define the Gauss formula. We also obtain Weingarten formula for the map using the linear connection $\nabla^{F\perp}$ in $(F_*(TM))^\perp$. From Gauss-Weingarten formula, we obtain Gauss, Ricci and Codazzi equations for submersion. The results below may be useful in the sequel.

Proposition 4.1. *Let $F : (M, g_M) \to (N, g_N)$ be a map between M^{5n+4} and N^{4n+3} for m odd and n then the following will equivalently hold:*

(i) *F is Riemannian at $p_1 \in TM$ and thus at every $p \in M$.*

(ii) *Π_{p_1} is a projection.*

(iii) *Π'_{p_1} is a projection.*

Proof. Since (M^{5n+4}, g_M) and (N^{4n+3}) are Riemannian manifolds, the map $F : M \to N$ is Riemannian map if there exists the adjoint map $*F_*$ of F_* characterized by:

$$g_M(X, *F_{*p_1}Y) = g_N(F_{*p_1}, Y)$$

for some $X \in T_{p_1}M$ and $Y \in T_{F(p_1)}N$ and $p_1 \in M$. Additionally, F is a smooth map between the manifolds M and N, thus we can define linear transformation:

$$\begin{aligned} \Pi_{p_1} : T_{p_1}M \to T_{p_1}M; \Pi_{p_1} &= *F_{*p_1} \circ F_{*p_1} \\ \Pi'_{p_1} : T_{p_2}N \to T_{p_2}N; \Pi'_{p_1} &= F_{*p_1} \circ *F_{*p_1}. \end{aligned}$$

Hence, $\Pi_{p_1} \circ \Pi_{p_1} = \Pi_{p_1}$ and $\Pi'_{p_1} \circ \Pi'_{p_1} = \Pi'_{p_1}$. So both Π_{p_1} and Π'_{p_1} are projections and the results above is completely characterized. □

5 Gauss-Weingarten Formulas for the Submersion between M and N

Let $F : M \to N$ be a smooth map between (M^{5n+4}, g_M) and (N^{4n+3}, g_N). Let $p_2 = F(p_1)$ for $p_1 \in M$. Suppose that $\overset{N}{\nabla}$ is a Levi-Civita connection on N, for $X \in \Gamma(TM)$ and $V \in \Gamma(TN)$, we have:

$$\overset{N}{\nabla} X(V \circ F) = \nabla^N_{F_*X} V \tag{5.1}$$

where $F^{-1}TN$ is the pullback bundle which has fibres $(F^{-1}TN)_p = T_{F(p)}N$ for $p \in M$. $Hom(TM, F^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. The second fundamental form of F is given by:

$$(\nabla F_*)(X,Y) = \overset{N}{\nabla^F}_X F_*(Y) - F_*(\nabla^M_X Y) \tag{5.2}$$

$X, Y \in \Gamma(TM)$. This form is symmetric. In addition $(\nabla F_*)(X,Y) \in \Gamma((kerF_*)^{\perp})$, for $X, Y \in \Gamma(TM)$, hence it lacks components in range F_*. The following results thus hold.

Proposition 5.1. *Let $F : M \to N$ be the submersion described. Then,*

$$g_N((\nabla F_*)(X,Y), (F_*(Z))) = 0 \tag{5.3}$$

For all, $X, Y, Z \in \Gamma(kerF_*)^{\perp}$

Proof. Clearly, $(\nabla F_*)(X,Y) \in \Gamma((rangeF_*)^{\perp}) \in \Gamma((kerF_*)^{\perp})$, for $X, Y, Z \in TM$. Thus at any $p \in M$, we write:

$$\overset{N}{\nabla^F}_X F_*(Y)(p) = F_*(\nabla^M_X Y)(p) + (\nabla F_*)(X,Y)(p) \tag{5.4}$$

for all $X, Y \in \Gamma(kerF_*)^{\perp}$ where $\overset{N}{\nabla^F}_X F_*(Y) \in T_F(p)N$, $F_*(\nabla^M_X Y)(p) \in F_{*p}(T_pM)$ and $(\nabla F_*)(X,Y)(p) \in (F_{*p}(T_pM))^{\perp}$. □

Let $F : M \to N$ be a Riemannian submersion, we define $\mathcal{T}$ and $\mathcal{A}$ as:

$$\mathcal{A}_E F = H\nabla^M_{HE}\mathcal{V}F + \mathcal{V}\nabla^M_{HE}HF \tag{5.5}$$

$$\mathcal{T}_E F = H\nabla^M_{VE}\mathcal{V}F + \mathcal{V}\nabla^M_{VE}HF \tag{5.6}$$

where $E, F \in M$ and ∇^M is the levi-civita connection on g_M.

From $TM = kerF_* \oplus H$, we see that, $\Pi_E = \mathcal{T}_{VE}$ and $\mathcal{A}_E = \mathcal{A}_{HE}$, hence $\mathcal{T}$ and $\mathcal{A}$ are vertical and horizontal respectively. Now $\mathcal{T}$ satisfies,

$$T_U W = T_W U$$

for all $U, W \in \Gamma(kerF_*)$. Again, from equation 5.5 and 5.6 we have:

$$\nabla^M_V W = \mathcal{T}_V W + \bar{\nabla}_V W \tag{5.7}$$

$$\nabla^M_V X = H\nabla^M_V X + \mathcal{T}_V X \tag{5.8}$$

$$\nabla^M_X V = A_X V + \mathcal{V}\nabla^M_X V \tag{5.9}$$

$$\nabla^M_X Y = H\nabla^M_X Y + \mathcal{A}_X Y \tag{5.10}$$

for all $X, Y \in \Gamma((kerF_*)^{\perp})$ and $V, W \in \Gamma(kerF_*)$ where $\bar{\nabla} = V\nabla^M_V W$. Let ∇^N denote both the levi-civita connection of (N, g_N) and its pullback along F. Then $\nabla^{F\perp}$ is a linear connection on $(F_*(TM))^{\perp}$ such that $\nabla^{F\perp} g_N = 0$.

Proposition 5.2. *Let $F : M \to N$ be a submersion. Then the map defined and denoted by S_V as:*

$$\nabla^N_{F_*X} V = -S_V F_* X + \nabla^{F\perp}_X V \tag{5.11}$$

where $S_V F_* X$ is the tangential component (a vector field along F) of $\nabla^N_{F_*X} V$ is symmetric linear transformation.

Proof. This has been obtained from the pullback connection of ∇^N, thus at $p_1 \in M$, we have: $\nabla^N_{F_*X}V(p_1) \in T_{F(p_1)}N$, $S_V F_* X(p_1) \in F_{*p_1}(T_{p_1}M)$ and $\nabla^{F\perp}_X V(p_1) \in (F_{*p_1}(T_{p_1}M)^{\perp})$. Clearly $S_V F_* X$ is biliniear in V and F_*X and $S_V F_* X$ at p_1 depend along on V_{p_1} and $F_{*p_1}X_{p_1}$. By direct computations, we obtain:

$$g_N(S_V F_* X, F_* Y) = g_N(V, (\nabla F_*)(X, Y)) \tag{5.12}$$

for $X, Y \in \Gamma(kerF_*)^{\perp}$ and $V \in \Gamma(rangeF_*)^{\perp}$. Since (∇F_*) is symmetric, it follows that S_V is a symmetric linear transformation of range F_*. □

Remark: The equations 5.1 is Gauss formula and equations 5.8, 5.9, 5.10 and 5.11 are weigharten equations for $F : M \to N$.

6 Gauss and Codazzi Equations for the Submersion F between M and N

Let $F : M \to N$ be a submersion, consider a linear transformation given and define by:

$F^{\lambda}_{*p_1} : (kerF_*)^{\perp}(p_1), g_{Mp_1}((kerF_*)^{\lambda}(p_1)) \to (rangeF_*(p_2), g_{Np_2}(rangeF_{*p_2})$

Denote the adjoint of F^{λ}_* by $*F^{\lambda}_*$ and by $*F_{*p_1}$ the adjoint of

$F_{*p_1} : (T_{p_1}M, g_{Mp_1}) \to (T_{p_2}N, g_{Np_2})$. Then the linear transformation:

$(*F_{*p_1})^{\lambda} : rangeF_*(p_2) \to (kerF_*)^{\perp}(p_1)$ defined by $(*F_{*P_1})^{\lambda}Y = *F_{*p_1}Y$ where $Y \in \Gamma(rangeF_{*p_1})$, $p_2 = F(p_1)$ is an isomorphism and $(F^{\lambda}_{*p_1})^{-1} = (*F_{*p_1})^{\lambda} = *(F^{\lambda}_{*p_1})$.

From equations 5.1 and 5.11 respectively we have:

$$\begin{aligned} R^N(F_*X, F_*Y)F_*Z &= -S_{(\nabla F_*)(Y,Z)}F_*X + S_{(\nabla F_*)(X,Z)}F_*Y \\ &+ F_*(R^M(X,Y)Z) + (\nabla_X(\nabla F_*))(Y,Z) \\ &- (\nabla_Y(\nabla F_*))(X,Z) \end{aligned} \tag{6.1}$$

for all $X, Y, Z \in \Gamma(kerF_*)^{\perp}$ where R^M, R^N denote the curvature tensor of ∇^M and ∇^N the metric connection on M and N. Moreover, $(\nabla_X(\nabla F_*))(Y, Z)$ is defined by:

$$\nabla_X(\nabla F_*)(Y, Z) = \nabla^{F\perp}_X(\nabla F_*)(Y, Z) - (\nabla F_*)(Y, \nabla^M_X Z) \tag{6.2}$$

From equation 6.1, for any vector $J \in \Gamma((kerF_*)^{\perp})$, we have:

$$\begin{aligned} g_N(R^N(F_*X, F_*Y)F_*Z, F_*J) &= g_M(R^M(X,Y)Z, J) \\ &+ g_N((\nabla F_*)(X,Z), (\nabla F_*)(Y,J)) \\ &- g_N((\nabla F_*)(Y,Z), (\nabla F_*)(X,J)). \end{aligned} \tag{6.3}$$

Taking the $\Gamma(rangeF^{\perp}_*)$ in equation 6.1 we have:

$$(R^N(F_*X, F_*Y)F_*Z)^{\perp} = (\nabla_X(\nabla F_*))(Y, Z) - (\nabla_Y(\nabla F_*))(X, Z). \tag{6.4}$$

The equations 6.1 and 6.3 are the Gauss and codazzi equations respectively for $F : M \to N$.
Next, let $X, Y \in TM$ and $V\perp \in \Gamma(rangeF_*)$, define the curvature tensor field $R^{F\perp}$ of the subbundle $(rangeF_*)^{\perp}$ by

$$R^{F\perp}(F_*(X), F_*(Y))V = \nabla^{F\perp}_X \nabla^{F\perp}_Y V - \nabla^{F\perp}_Y \nabla^{F\perp}_X V - \nabla_{[X,Y]}{}^{F\perp} \tag{6.5}$$

Then using Gauss-Weingarten equation 5.12, we obtain:

$$\begin{aligned} R^N(F_*(X),F_*(Y))V &= R^{F\perp}(F_*(X),F_*(Y))V - F_*(\nabla^M_X * F_*(S_V F_*(Y))) \\ &+ S_{\nabla^{F\perp}_X V} F_*(Y) + F_*(\nabla^M_Y * F_*(S_V F_*(X))) \\ &- S_{\nabla^{F\perp}_X V} F_*(X) - (\nabla F_*)(X, *F_*(S_V F_*(Y))) \\ &+ (\nabla F_*)(Y, *F_*(S_V F_*(X))) - S_V F_*([X,Y]) \end{aligned} \tag{6.6}$$

where,

$$F_*([X,Y]) = \overset{N}{\nabla^F}{}_X F_*(Y) - \overset{N}{\nabla^F}{}_Y F_*(X).$$

Then for $F_*(Z) \in \Gamma(rangeF_*)$, we have:

$$\begin{aligned} g_N(R^N(F_*(X),F_*(Y))V, F_*(Z)) &= g_N((\tilde{\nabla}_Y S)_V F_*(X), F_*(Z)) \\ &- g_N((\tilde{\nabla}_X S)_V F_*(Y), F_*(Z)) \end{aligned} \tag{6.7}$$

where,

$$(\tilde{\nabla}_X S)_V F_*(Y) = F_*(\nabla^M_X * F_*(S_V F_*(Y))) - S_{\nabla^{F\perp}_X V} F_*(Y) - S_V \Pi \overset{N}{\nabla^F}{}_X F_*(Y)$$

where Π denotes the projection morphism on the range F_*. On the other hand, for $W \in \Gamma(rangeF_*^{\perp})$, we get,

$$\begin{aligned} g_N(R^N(F_*(X),F_*(Y))V, W) &= g_N(R^{F\perp}(F_*(X),F_*(Y))V, W) - g_N((\nabla F_*)(X, *F_*(S_V F_*(Y))), W) \\ &+ g_N((\nabla F_*)(Y, *F_*(S_V F_*(X))), W) \end{aligned} \tag{6.8}$$

Using Gauss-Weingarten equation 5.12 , we obtain:

$$g_N((\nabla F_*)(X, *F_*(S_V F_*(Y))), W) = g_N(S_W F_*(X), S_V F_*(Y)) \tag{6.9}$$

Since S_V is self adjoint, we get:

$$g_N((\nabla F_*)(X, *F_*(S_V F_*(Y))), W) = g_N(S_V S_W F_*(X), F_*(Y) \tag{6.10}$$

using equation 6.9 and 6.10 we arrive at:

$$\begin{aligned} g_N(R^N(F_*(X),F_*(Y))V, W) &= g_N(R^{F\perp}(F_*(X),F_*(Y))V, W) \\ &+ g_N([S_W, S_V]F_*(X), F_*(Y)) \end{aligned} \tag{6.11}$$

where $[S_W, S_V] = S_W S_V - S_V S_W$. The last equation 6.11 is the Ricci equation for submersion $F : M \to N$.

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Competing Interests

Authors have declared that no competing interests exist.

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